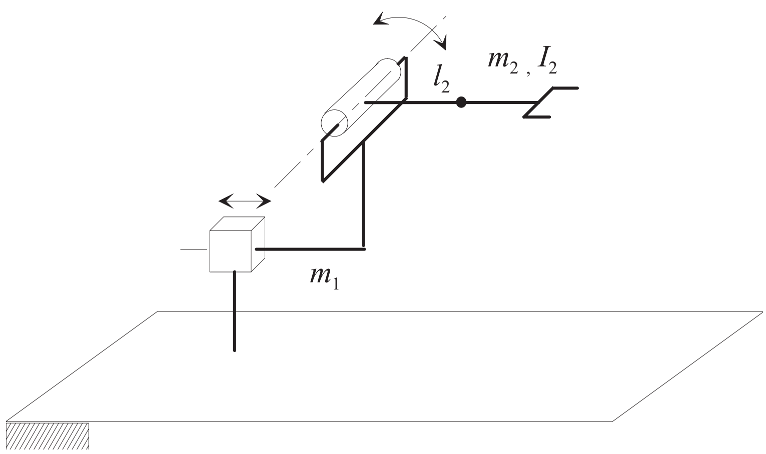
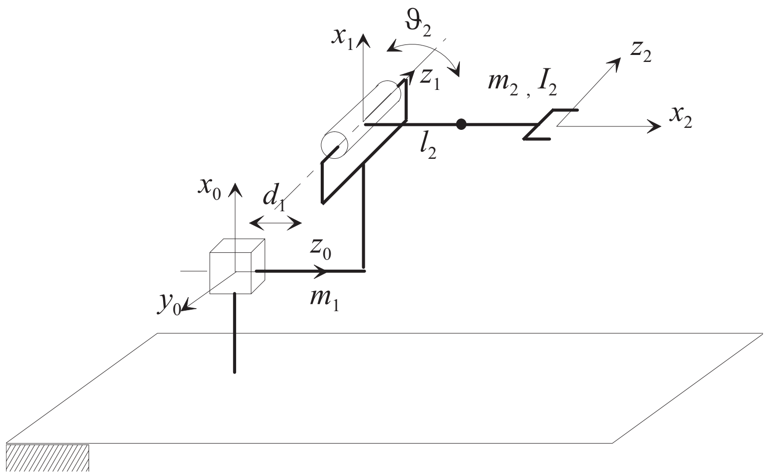
EXERCISE 1

1. Consider the manipulator sketched in the picture:



Find the expression of the inertia matrix $\mathbf{B}(\mathbf{q})$ of the manipulator[1]

Denavit-Hartenberg frames can be defined as sketched in this picture:



Computations of the Jacobians:

Link 1

$$\mathbf{J}_P^{(l_1)} = \left[\begin{array}{cc} \mathbf{j}_{P_1}^{(l_1)} & \mathbf{0} \end{array} \right] = \left[\begin{array}{cc} \mathbf{z}_0 & \mathbf{0} \end{array} \right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

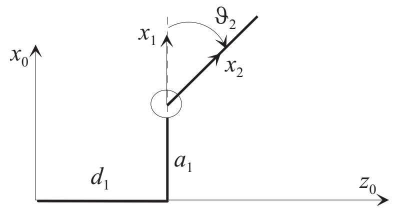
Link 2

$$\mathbf{J}_P^{(l_2)} = \left[\begin{array}{cc} \mathbf{j}_{P_1}^{(l_2)} & \mathbf{j}_{P_2}^{(l_2)} \end{array} \right] = \left[\begin{array}{cc} \mathbf{z}_0 & \mathbf{z}_1 \times (\mathbf{p}_{l_2} - \mathbf{p}_1) \end{array} \right] = \begin{bmatrix} 0 & -l_2 s_2 \\ 0 & 0 \\ 1 & l_2 c_2 \end{bmatrix}$$

[1]The cross product between vector $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is $c = a \times b = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$

$$\mathbf{J}_O^{(l_2)} = \left[\begin{array}{cc} \mathbf{j}_{O_1}^{(l_2)} & \mathbf{j}_{O_2}^{(l_2)} \end{array} \right] = \left[\begin{array}{cc} \mathbf{0} & \mathbf{z}_1 \end{array} \right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{array} \right]$$

For the above computations, we can make reference to the following picture:



and to the following auxiliary vectors:

$$\mathbf{p}_{l_2} = \left[\begin{array}{c} a_1 + l_2 c_2 \\ 0 \\ d_1 + l_2 s_2 \end{array} \right], \mathbf{p}_1 = \left[\begin{array}{c} a_1 \\ 0 \\ d_1 \end{array} \right], \mathbf{z}_1 = \left[\begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right]$$

The inertia matrix can be computed now:

$$\begin{aligned} \mathbf{B}(\mathbf{q}) &= m_1 \mathbf{J}_P^{(l_1)^T} \mathbf{J}_P^{(l_1)} + m_2 \mathbf{J}_P^{(l_2)^T} \mathbf{J}_P^{(l_2)} + I_2 \mathbf{J}_O^{(l_2)^T} \mathbf{J}_O^{(l_2)} \\ &= m_1 \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] + m_2 \left[\begin{array}{cc} 1 & l_2 c_2 \\ l_2 c_2 & l_2^2 \end{array} \right] + I_2 \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \\ &= \left[\begin{array}{cc} b_{11} & b_{12} \\ b_{12} & b_{22} \end{array} \right] \end{aligned}$$

where:

$$\begin{aligned} b_{11} &= m_1 + m_2 \\ b_{12} &= m_2 l_2 c_2 \\ b_{22} &= m_2 l_2^2 + I_2 \end{aligned}$$

2. Ignoring the Coriolis and centrifugal terms, write the dynamic model of the manipulator.

Since the vertical axis is the $\mathbf{x}_0$ axis pointing upwards, the gravity acceleration vector is:

$$\mathbf{g}_0 = \left[\begin{array}{c} -g \\ 0 \\ 0 \end{array} \right]$$

The gravitational torques are thus:

$$\mathbf{g} = \left[\begin{array}{c} g_1 \\ g_2 \end{array} \right],$$

where:

$$g_1 = -m_1 \mathbf{g}_0^T \mathbf{j}_{P_1}^{(l_1)} - m_2 \mathbf{g}_0^T \mathbf{j}_{P_1}^{(l_2)} = -m_1 \mathbf{g}_0^T \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] - m_2 \mathbf{g}_0^T \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right] = 0$$

$$g_2 = -m_1 \mathbf{g}_0^T \mathbf{j}_{P_2}^{(l_1)} - m_2 \mathbf{g}_0^T \mathbf{j}_{P_2}^{(l_2)} = -m_2 \mathbf{g}_0^T \left[\begin{array}{c} -l_2 s_2 \\ 0 \\ l_2 c_2 \end{array} \right] = -m_2 g l_2 s_2$$

Neglecting Coriolis and centrifugal terms, the dynamic model can be written as:

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau$$

The two equations that form the model are:

$$\begin{aligned} (m_1 + m_2)\,\ddot{d}_1 + m_2 l_2 c_2 \ddot{\vartheta}_2 &= \tau_1 \\ m_2 l_2 c_2 \ddot{d}_1 + \left(m_2 l_2^2 + I_2\right)\ddot{\vartheta}_2 - m_2 g l_2 s_2 &= \tau_2 \end{aligned}$$

3. Show that the dynamic model is linear with respect to a certain set of dynamic parameters.

   The model can be written in the following form which is linear in the dynamic parameters:

$$\mathbf{Y}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right)\mathbf{\Pi} = \tau$$

   where the vector of dynamic parameters is expressed as:

$$\mathbf{\Pi} = \begin{bmatrix} m_1 + m_2 \\ m_2 l_2 \\ m_2 l_2^2 + I_2 \end{bmatrix}$$

   while the regressor matrix is:

$$\mathbf{Y} = \begin{bmatrix} \ddot{d}_1 & c_2\ddot{\vartheta}_2 & 0 \\ 0 & c_2\ddot{d}_1 - g s_2 & \ddot{\vartheta}_2 \end{bmatrix}$$

4. Write the expression of a "PD + gravity compensation" control law in the joint space for this specific manipulator.

   The vector equation of the control law is:

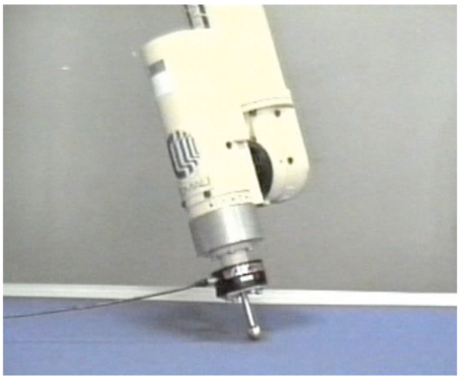
$$\tau = \mathbf{K}_P\left(\mathbf{q}_d - \mathbf{q}\right) - \mathbf{K}_P\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q})$$

   and corresponds, for the given manipulator, to the following two equations:

$$\begin{aligned} \tau_1 &= K_{P_1}(q_{d1} - q_1) - K_{D_1}\dot{q}_1 \\ \tau_2 &= K_{P_2}(q_{d2} - q_2) - K_{D_2}\dot{q}_2 - m_2 g l_2 s_2 \end{aligned}$$

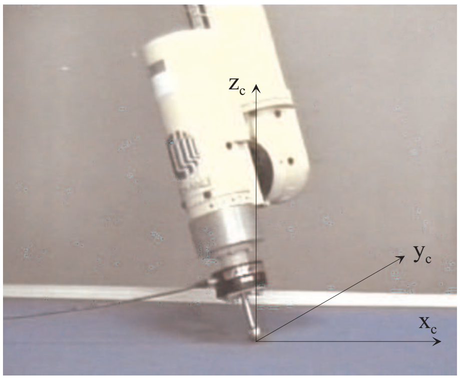
EXERCISE 2

1. Consider an interaction task of a manipulator, with a frictionless and rigid surface, as in this picture:



Assume a point contact and draw a contact frame directly on the picture. Based on this frame and neglecting angular velocities and moments, express the natural and the artificial constraints for this problem, and specify the selection matrix.

The contact frame can be conveniently chosen as in the following picture:



The natural constraints and artificial constraints can be easily identified:

Natural constraints	Artificial constraints
f_x^c	$\dot{p}_x^c$
f_y^c	$\dot{p}_y^c$
$\dot{p}_z^c$	f_z^c

The selection matrix is thus:

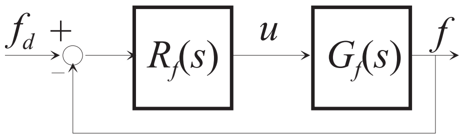
$$\Sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Explain what an implicit force controller is and why it might be convenient with respect to an explicit solution.

An implicit force control is closed around the position control loops. This is usually the only viable solution to implement force control, since the reliable and industrially safe position controllers cannot be bypassed.

3. Suppose now that along the force controlled direction an explicit force controller has to be designed. Sketch the block diagram of such controller and design it taking a bandwidth of 30 rad/s.

The block diagram of an explicit force controller in case of rigid surface is sketched in the picture:



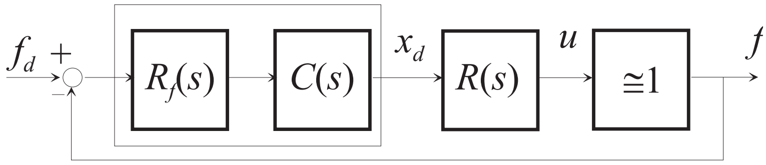
where the transfer function $G_f(s)$ is practically a unitary gain. We can then consider as a force controller an integrator:

$$R_f(s) = \frac{k_{if}}{s}$$

and the gain can be set equal to the desired bandwidth: $k_{if} = 30$.

4. Repeat the process in case an implicit force controller, for the same bandwidth, has to be designed.

   The block diagram of an implicit force controller in case of rigid surface is sketched in the picture:



   where $R(s)$ is the transfer function of the position controller. If we assume a PID position controller:

$$R(s) = \frac{K_D s^2 + K_P s + K_I}{s}$$

   The partial compensator of such controller is:

$$C(s) = \frac{1}{K_D s^2 + K_P s + K_I}$$

   If we select a PI controller on the force error:

$$R_f(s) = k_{pf} + \frac{k_{if}}{s}$$

   the loop transfer function becomes:

$$L_f(s) = \frac{s k_{pf} + k_{if}}{s^2}$$

   Since the high frequency approximation of such transfer function is k_{pf}/s we can set $k_{pf} = 30$ (equal to the required bandwidth. The zero of the controller can be set at a lower frequency range, for example $k_{if}/k_{pf} = 3$, which yields $k_{if} = 90$.

EXERCISE 3

1. Consider a wheel rolling without slipping on the horizontal plane, keeping the sagittal plane in the vertical direction. Write the expression of the pure rolling constraint in the case of a steerable wheel, explaining its physical meaning.

   The pure rolling constraint has always the same form, independently of the fact that the wheel is fixed or steerable, and is given by

$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0$$

   where x, y are the positions of the wheel contact point with respect to a fixed reference frame and θ is the wheel sagital plane orientation.

   This constraint means that the resultant of the velocities perpendicular to the direction of motion (the intersection between the sagittal and the motion plane) is equal to zero.

2. Show that the previous kinematic constraint is a nonholonomic constraint.

   Rewriting the constraint in Pfaffian form

$$\begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0$$

we can show it is nonholonomic using the necessary and sufficient condition. If the constraint were holonomic, we should find a function $\alpha(\mathbf{q})$ that satisfies the following equations

$$\frac{\partial(\alpha(\mathbf{q})\sin\theta)}{\partial y} = -\frac{\partial(\alpha(\mathbf{q})\cos\theta)}{\partial x} \tag{1}$$

$$\frac{\partial(\alpha(\mathbf{q})\sin\theta)}{\partial\theta} = 0 \tag{2}$$

$$0 = -\frac{\partial(\alpha(\mathbf{q})\cos\theta)}{\partial\theta} \tag{3}$$

From equation (1) we get

$$\sin\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = -\cos\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

from (2)

$$\sin\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta} + \alpha(\mathbf{q})\cos\theta = 0 \qquad \Rightarrow \qquad -\cos\theta = \frac{\sin\theta}{\alpha(\mathbf{q})}\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}$$

and from (3)

$$-\cos\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta} + \alpha(\mathbf{q})\sin\theta = 0 \qquad \Rightarrow \qquad \sin\theta = \frac{\cos\theta}{\alpha(\mathbf{q})}\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}$$

Substituting the previous two expressions in equation (1) we get

$$\frac{\cos\theta}{\cancel{\alpha(\mathbf{q})}}\cancel{\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}}\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = \frac{\sin\theta}{\cancel{\alpha(\mathbf{q})}}\cancel{\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}}\frac{\partial(\alpha(\mathbf{q}))}{\partial x} \qquad \Rightarrow \qquad \cos\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = \sin\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

or, equivalently

$$\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = \tan\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

However, from equation (1) we also get

$$\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = -\frac{1}{\tan\theta}\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

These two conditions can be satisfied at the same time if and only if $\alpha(\mathbf{q}) = 0$. We thus conclude that the constraint is nonholonomic.

3. Consider now the dynamics of a steerable rolling wheel, describe the two most important modelling approaches that can be used to represent the wheel-ground interaction (longitudinal and lateral) forces stressing their differences and/or similarities.

   There are two main approaches to model the wheel-ground interaction.

   The first one is the empirical approach. In empirical tire models an experimental dataset including lateral forces and corresponding slip angles is assumed to be available, and a class of mathematical functions suitable to fit the dataset is selected. The solution of the fitting problem represents the tire model. A classical example of fitting function is the Pacejka Magic Forrmula.

   The second one is the physical approach. In this case the model that explains the force-slip relation is derived using physical principles. An example of physical model is the brush or Fiala model.

4. Describe a linear model to represent the wheel lateral force.

   A linear model, that holds for small slip angles, is $F_x = C_x\sigma_x$ where C_x is the cornering stiffness and σ_x the lateral slip.

EXERCISE 4

1. Consider a robot represented by a bicycle kinematic model. Describe an algorithm that allows to find a trajectory (only the expressions of $x(t)$ and $y(t)$ are required), feasible with respect to the kinematic model, to move the robot in an obstacle free environment from an initial state $q_i = [x_i \; y_i \; \theta_i \; v_i]$ at $t_i = 0$ to a final state $q_f = [x_f \; y_f \; \theta_f \; v_f]$ at $t_f = \bar{t}_f$ (where the value of $\bar{t}_f$ is known), exploiting the flatness property.

We consider the following simplified bicycle model

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta \qquad \dot{\theta} = \frac{v}{\ell}\tan\phi$$

we know it is flat with respect to the flat outputs $z_1 = x$ and $z_2 = y$.
The initial and the final configurations allow to enforce 8 constraints, we can thus select for z_1 and z_2 the following two third order polynomials

$$z_1(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \qquad z_2(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$$

characterized by 8 coefficients that can be determined imposing the initial and final conditions

$$\begin{aligned}
&z_1(0) = a_0 = x_i && \dot{z}_1(0) = a_1 = v_i\cos(\theta_i)\\
&z_2(0) = b_0 = y_i && \dot{z}_2(0) = b_1 = v_i\sin(\theta_i)\\
&z_1(\bar{t}_f) = a_0 + a_1\bar{t}_f + a_2\bar{t}_f^2 + a_3\bar{t}_f^3 = x_f && \dot{z}_1(\bar{t}_f) = a_1 + 2a_2\bar{t}_f + 3a_3\bar{t}_f^2 = v_f\cos(\theta_f)\\
&z_2(\bar{t}_f) = b_0 + b_1\bar{t}_f + b_2\bar{t}_f^2 + b_3\bar{t}_f^3 = y_f && \dot{z}_2(\bar{t}_f) = b_1 + 2b_2\bar{t}_f + 3b_3\bar{t}_f^2 = v_f\sin(\theta_f)
\end{aligned}$$

Summarising, given an initial and a final configuration we can compute the a_i and b_i coefficients as

$$a_0 = x_i \qquad a_1 = v_i\cos(\theta_i) \qquad b_0 = y_i \qquad b_1 = v_i\sin(\theta_i)$$

and

$$\begin{bmatrix} a_2\\ a_3\\ b_2\\ b_3 \end{bmatrix} = \left(\begin{bmatrix} \bar{t}_f^2 & \bar{t}_f^3 & 0 & 0\\ 0 & 0 & \bar{t}_f^2 & \bar{t}_f^3\\ 2\bar{t}_f & 3\bar{t}_f^2 & 0 & 0\\ 0 & 0 & 2\bar{t}_f & 3\bar{t}_f^2 \end{bmatrix}\right)^{-1} \begin{bmatrix} x_f - a_0 - a_1\bar{t}_f\\ y_f - b_0 - b_1\bar{t}_f\\ v_f\cos(\theta_f) - a_1\\ v_f\sin(\theta_f) - b_1 \end{bmatrix}$$

Then the robot trajectory is given by

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \qquad y(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$$

2. How can the same planning problem be solved, considering obstacles as well, using a sampling-based planning algorithm?

A sampling-based planner, like for example RRT*, in its kinodynamic version allows to solve the planning problem, considering obstacles and enforcing the satisfaction of the kinematic model.

This can be achieved introducing a steering function that compute an edge solving a TPBV problem like the following one

$$\min_{a(t),\phi(t),\tau} \int_0^{\tau} \left(1 + \left[\phi(t) \;\; a(t)\right] R \left[\phi(t) \;\; a(t)\right]^T\right) \mathrm{d}t$$

$$\begin{aligned}
\dot{x} &= v\cos\theta\\
\dot{y} &= v\sin\theta\\
\dot{\theta} &= \frac{v}{\ell}\tan\phi\\
\dot{v} &= a
\end{aligned}$$

$$\begin{aligned}
&x(0) = x_i,\; y(0) = y_i,\; \theta(0) = \theta_i,\; v(0) = v_i\\
&x(\tau) = x_f,\; y(\tau) = y_f,\; \theta(\tau) = \theta_f,\; v(\tau) = v_f
\end{aligned}$$

where the cost function can be selected in accordance with the requirements of the considered planning problem.

3. Consider the following bicycle kinematic model

$$\dot{x} = v \cos\theta \qquad \dot{y} = v \sin\theta \qquad \dot{\theta} = \frac{v}{\ell} \tan\phi$$

and a point P related to the bicycle rear wheel contact point (x, y) by the following relations

$$x_P = x + \varepsilon \cos\theta \qquad y_P = y + \varepsilon \sin\theta$$

Show how a feedback control law that linearises the bicycle model can be derived.

The feedback linearising control law derived for the unicycle kinematic model

$$\begin{aligned} v &= v_{xP} \cos(\theta) + v_{yP} \sin(\theta) \\ \omega &= \frac{v_{yP} \cos(\theta) - v_{xP} \sin(\theta)}{\varepsilon} \end{aligned}$$

where v_{xP}, v_{yP}, are the velocities of point P, can be used for the bicycle kinematic model as well, setting $\omega = \frac{v}{\ell} \tan\phi$.

We thus obtain the following linearising control law

$$\begin{aligned} v &= v_{xP} \cos(\theta) + v_{yP} \sin(\theta) \\ \phi &= \arctan\left(\frac{\ell \, v_{yP} \cos(\theta) - v_{xP} \sin(\theta)}{\varepsilon \, v_{xP} \cos(\theta) + v_{yP} \sin(\theta)} \right) \end{aligned}$$

4. Explain why the control system designed in the previous step cannot be used to regulate the pose of the robot.

Due to the feedback linearising controller the orientation of the robot becomes an hidden state, and cannot be controlled any more. For this reason that control system cannot be used to control the robot heading but only its position.