Automatic Control
Exercise 2: Time responses of first/second order systems
Prof. Luca Bascetta

Exercise 1

Given the following continuous time linear and time invariant dynamical system

$$\begin{aligned} \dot{x}(t) &= x(t-\tau) + u(t) \\ y(t) &= x(t) \end{aligned}$$

with $\tau > 0$, compute the transfer function.

Solution

Assuming zero initial conditions and applying the Laplace transform to the state and output equations we obtain

$$\begin{aligned} sX(s) &= X(s)e^{-s\tau} + U(s) \\ Y(s) &= X(s) \end{aligned}$$

Solving now the first equation with respect to $X(s)$

$$\left(s - e^{-s\tau}\right) X(s) = U(s) \quad \Rightarrow \quad X(s) = \frac{U(s)}{s - e^{-s\tau}}$$

and substituting into the output equation

$$Y(s) = \frac{U(s)}{s - e^{-s\tau}}$$

The transfer function of the LTI system is thus

$$G(s) = \frac{1}{s - e^{-s\tau}}$$

Exercise 2

Consider a continuous time linear and time invariant dynamical system whose step response is characterized by

$$\begin{aligned} y(0) = \dot{y}(0) = \ddot{y}(0) &= 0 \\ \dddot{y}(0) &= -4 \\ \lim_{t \to \infty} y(t) &= -32 \end{aligned}$$

Find the system transfer function.

Solution

Assuming that $G(s)$ is the system transfer function, the step response in the Laplace domain is

$$Y(s) = \frac{G(s)}{s}$$

and applying the initial value theorem we obtain

$$\begin{aligned} y(0) &= \lim_{s \to \infty} sY(s) = \lim_{s \to \infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s \to \infty} s^2 Y(s) = \lim_{s \to \infty} sG(s) = 0 \\ \ddot{y}(0) &= \lim_{s \to \infty} s^3 Y(s) = \lim_{s \to \infty} s^2 G(s) = 0 \\ \dddot{y}(0) &= \lim_{s \to \infty} s^4 Y(s) = \lim_{s \to \infty} s^3 G(s) = -4 \end{aligned}$$

A transfer function that satisfies the previous constraints is

$$G(s) = -\frac{1}{0.25s^3 + bs^2 + cs + d}$$

Applying now the final value theorem

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} G(s) = -32 \quad \Rightarrow \quad -\frac{1}{d} = -32$$

Finally, a transfer function that satisfies all the constraints is

$$G(s) = -\frac{32}{8s^3 + 32s^2 + 32s + 1}$$

In order to be sure that the final value theorem is applicable, we have to verify that the system is asymptotically stable. Using the Routh criterion

$$\begin{array}{ccc} 8 & 32 & 0 \\ 32 & 1 & 0 \\ 31.75 & 0 & \\ 1 & & \end{array}$$

we conclude that the system is asymptotically stable.

Exercise 3

Compute the response of the system

$$G(s) = \frac{s-1}{(s+1)(s+2)}$$

to the input $u(t) = e^{-\alpha t}$, $\alpha \in \mathbb{R}$.
In particular, compute the initial value and, whenever possible, the final value of the response, and its analytical expression.

Solution

The system output in the Laplace domain is given by

$$Y(s) = \frac{s-1}{(s+1)(s+2)(s+\alpha)}$$

Applying the initial value theorem we obtain

$$y(0) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{s(s-1)}{(s+1)(s+2)(s+\alpha)} = 0$$

and from the final value theorem, assuming $\alpha \geq 0$ or $\alpha = -1$, we obtain

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{s(s-1)}{(s+1)(s+2)(s+\alpha)} = \begin{cases} 0 & \alpha > 0 \text{ or } \alpha = -1 \\ -\dfrac{1}{2} & \alpha = 0 \end{cases}$$

Assuming $\alpha = -1$ and applying the Heaviside method we obtain

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+2} = \frac{\alpha_1(s+2) + \alpha_2(s+1)}{(s+1)(s+2)}$$

Evaluating the numerator for $s = -2$ and $s = -1$ we obtain $\alpha_1 = 1$ and $\alpha_2 = -1$, and the Laplace transform of the output becomes

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Applying now the inverse Laplace transform we obtain the analytical expression of the output response

$$y(t) = e^{-t} - e^{-2t} \quad t \geq 0$$

Assuming $\alpha = 1$ and applying the Heaviside method we obtain

$$Y(s) = \frac{s-1}{(s+1)^2(s+2)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{(s+1)^2} + \frac{\alpha_3}{s+2} = \frac{\alpha_1(s+1)(s+2) + \alpha_2(s+2) + \alpha_3(s+1)^2}{(s+1)^2(s+2)}$$

From straightforward calculations we conclude that $\alpha_1 = 3$, $\alpha_2 = -2$, and $\alpha_3 = -3$, and the Laplace transform of the output becomes

$$Y(s) = \frac{3}{s+1} - \frac{2}{(s+1)^2} - \frac{3}{s+2}$$

Applying now the inverse Laplace transform we obtain the analytical expression of the output response

$$y(t) = 3e^{-t} - 2te^{-t} - 3e^{-2t} \quad t \geq 0$$

Assuming $\alpha = 2$ and applying the Heaviside method we obtain

$$Y(s) = \frac{s-1}{(s+1)(s+2)^2} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+2} + \frac{\alpha_3}{(s+2)^2} = \frac{\alpha_1(s+2)^2 + \alpha_2(s+1)(s+2) + \alpha_3(s+1)}{(s+1)(s+2)^2}$$

From straightforward calculations we conclude that $\alpha_1 = -2$, $\alpha_2 = 2$, and $\alpha_3 = 3$, and the Laplace transform of the output becomes

$$Y(s) = -\frac{2}{s+1} + \frac{2}{s+2} + \frac{3}{(s+2)^2}$$

Applying now the inverse Laplace transform we obtain the analytical expression of the output response

$$y(t) = -2e^{-t} + 2e^{-2t} + 3te^{-2t} \quad t \geq 0$$

For all the other values, applying the Heaviside method we obtain

$$Y(s) = \frac{s-1}{(s+1)(s+2)(s+\alpha)} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+2} + \frac{\alpha_3}{s+\alpha} = \frac{\alpha_1(s+2)(s+\alpha) + \alpha_2(s+1)(s+\alpha) + \alpha_3(s+1)(s+2)}{(s+1)(s+2)(s+\alpha)}$$

From straightforward calculations we conclude that

$$\alpha_1 = \frac{2}{1-\alpha} \qquad \alpha_2 = \frac{3}{\alpha-2} \qquad \alpha_3 = \frac{\alpha+1}{(\alpha-1)(2-\alpha)}$$

and the Laplace transform of the output becomes

$$Y(s) = \frac{2}{1-\alpha}\frac{1}{s+1} + \frac{3}{\alpha-2}\frac{1}{s+2} + \frac{\alpha+1}{(\alpha-1)(2-\alpha)}\frac{1}{s+\alpha}$$

Applying now the inverse Laplace transform we obtain the analytical expression of the output response

$$y(t) = \frac{2}{1-\alpha}e^{-t} + \frac{3}{\alpha-2}e^{-2t} + \frac{\alpha+1}{(\alpha-1)(2-\alpha)}e^{-\alpha t} \quad t \geq 0$$

Exercise 4

Consider the following transfer functions

$$G_1(s) = \frac{s+10}{s^2+2s+4} \qquad G_2(s) = \frac{1-0.1s}{1+7s+10s^2}$$

Compute the gain-time constant form of $G_1(s)$, and the zero-pole form of $G_2(s)$.

Solution

Analysing function $G_1(s)$ we obtain

$$\mu_1 = G_1(0) = \frac{10}{4} = \frac{5}{2}$$

and we observe that there are two complex and conjugate poles with natural frequency $\omega_n = 2$ and damping $\xi = 0.5$.
The gain-time constant form is thus

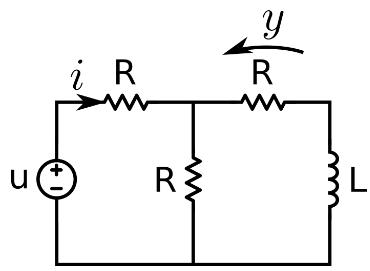
$$G_1(s) = \frac{5}{2}\frac{1+0.1s}{1+0.5s+0.25s^2}$$

From transfer function $G_2(s)$, instead, we obtain

$$G_2(s) = \frac{1-0.1s}{1+7s+10s^2} = -0.01\frac{s-10}{s^2+0.7s+0.1} = -0.01\frac{s-10}{(s+0.5)(s+0.2)}$$

Exercise 6

Consider the following electric circuit



Write the state space equations of the dynamical system that describes the circuit and compute the step response.

Solution

Considering the voltage equation for two loops we obtain

$$u(t) = Ri(t) + L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} + Ri_L(t)$$
$$u(t) = Ri(t) + R\left(i(t) - i_L(t)\right)$$

Solving the second equation with respect to i and substituting into the first one we obtain

$$L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} = u(t) - Ri_L(t) - \frac{u(t)}{2} - \frac{R}{2}i_L(t)$$

and assuming $x(t) = i_L(t)$ the equations of the dynamical system are

$$\dot{x}(t) = -\frac{3R}{2L}x(t) + \frac{1}{2L}u(t)$$
$$y(t) = Rx(t)$$

We can now transform the state equation obtaining

$$(2Ls + 3R)\,X(s) = U(s) \quad \Rightarrow \quad X(s) = \frac{U(s)}{3R + 2Ls}$$

The transfer function of the system is thus

$$G(s) = \frac{1}{3}\frac{1}{1 + s\dfrac{2L}{3R}}$$

Fig. 1 shows the step response of this transfer function for $L = 3$ and $R = 2$.

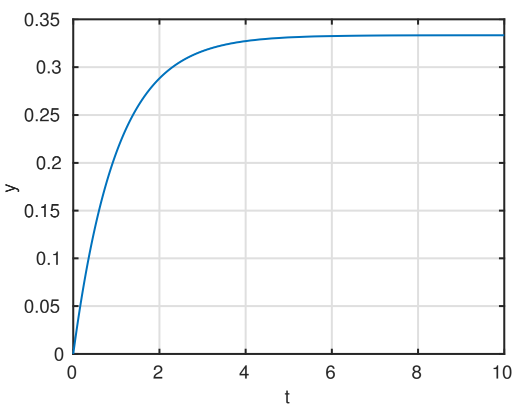


Figure 1: Step response for $L = 3$ and $R = 2$.

Exercise 7

Given the following transfer function

$$G(s) = \frac{10}{(1+s)(1+100s)}$$

Compute the step response and its analytic expression.
Write a first order approximation of $G(s)$.

Solution

Being the system asymptotically stable, we can apply the initial and final value theorems, obtaining

$$\begin{aligned} y(0) &= \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} G(s) = 0 \\ \dot{y}(0) &= \lim_{s\to\infty} s^2Y(s) = \lim_{s\to\infty} sG(s) = 0 \\ \ddot{y}(0) &= \lim_{s\to\infty} s^3Y(s) = \lim_{s\to\infty} s^2G(s) = \frac{1}{10} \\ \lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) = 10 \end{aligned}$$

Considering that the transfer function is characterised by two time constants, $T_1 = 1$ and $T_2 = 100$, with one that is definitely bigger than the other, the step response can be approximated with the response of a first order system characterised by the slower dynamic.
Moreover, a first order approximation of $G(s)$ is given by

$$\tilde{G}(s) = \frac{10}{1+100s}$$

Figs. 2 and 3 show the step response of $G(s)$ and of $\tilde{G}(s)$.

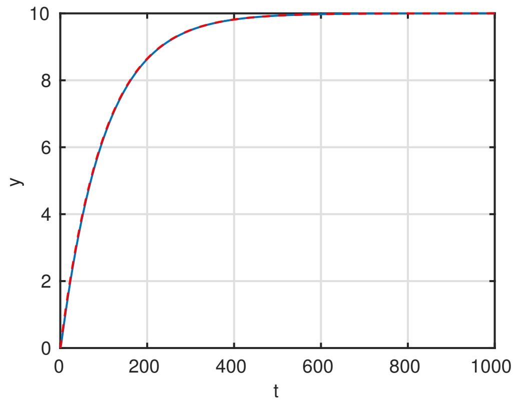


Figure 2: Step response of $G(s)$ (blue line) and of $\tilde{G}(s)$ (red dashed line).

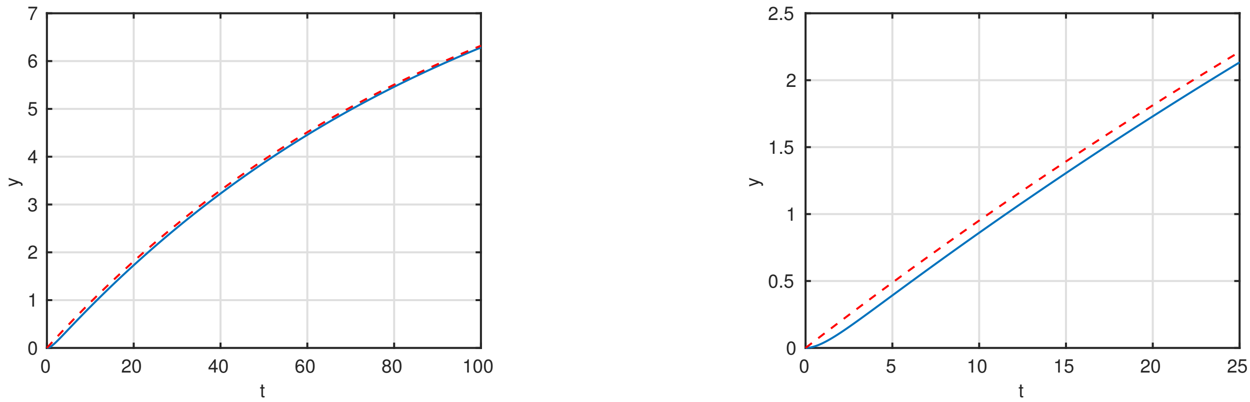


Figure 3: Initial transient of the step response of $G(s)$ (blue line) and of $\tilde{G}(s)$ (red dashed line).

Using the Heaviside method the Laplace transform of the step response can be decomposed as follows

$$Y(s) = \frac{0.1}{s(s+1)(s+0.01)} = \frac{\alpha}{s} + \frac{\beta}{s+1} + \frac{\gamma}{s+0.01} = \frac{\alpha(s+1)(s+0.01) + \beta s(s+0.01) + \gamma s(s+1)}{s(s+1)(s+0.01)}$$

Enforcing the equivalence of the numerators for $s = 0$, $s = -1$ and $s = -0.01$, we obtain $\alpha = 10$, $\beta = \frac{10}{99}$ and $\gamma = -\frac{1000}{99}$.
The step response can be thus decomposed as

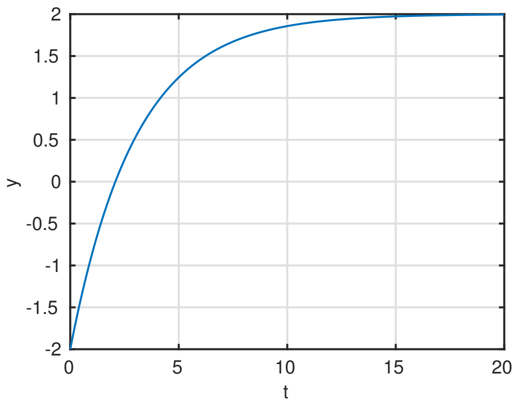
$$Y(s) = \frac{10}{s} + \frac{10}{99}\frac{1}{s+1} - \frac{1000}{99}\frac{\gamma}{s+0.01}$$

and applying the inverse Laplace transform we obtain

$$y(t) = 10 + \frac{10}{99}e^{-t} - \frac{1000}{99}e^{-0.01t} \quad t \geq 0$$

Exercise 8

Determine a transfer function that has the following step response



Solution

A non-minimum phase first order system, i.e., with a zero in the right half plane, has a step response with the characteristics shown in the figure. We thus consider the following transfer function

$$G(s) = \mu\frac{1 - s\tau}{1 + sT} \qquad T > 0 \text{ and } \tau > 0$$

Applying the initial and final value theorems we obtain

$$y(0) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\mu\frac{1-s\tau}{1+sT}\frac{1}{s} = -\frac{\mu\tau}{T}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\mu\frac{1-s\tau}{1+sT}\frac{1}{s} = \mu$$

Analysing the figure we discover that

$$\mu = 2 \qquad T = \frac{15}{5} = 3$$

and

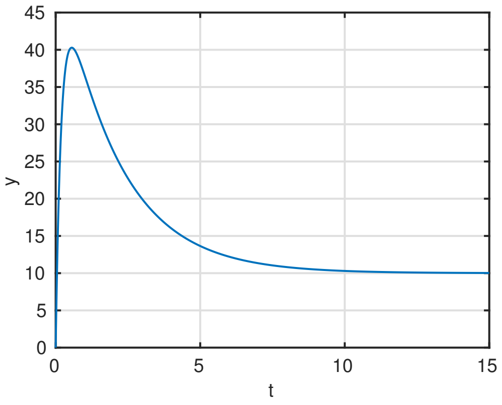
$$-\frac{\mu\tau}{T} = -2 \Rightarrow \tau = 3$$

The transfer function is thus

$$G(s) = 2\frac{1-3s}{1+3s}$$

Exercise 9

Given the following step response



Find, among the following transfer functions

$$G_1(s) = \frac{10}{(1+2s)(1+0.2s)} \qquad G_2(s) = 10\frac{1-4s}{(1+2s)(1+0.2s)}$$
$$G_3(s) = 10\frac{1+0.1s}{(1+2s)(1+0.2s)} \qquad G_4(s) = 10\frac{1+10s}{(1+2s)(1+0.2s)}$$

the one that can have the step response in the figure.

Solution

Analysing the transfer functions we discover that:

- $G_1(s)$ has no zeros, the step response is characterised by $\dot{y}(0) = 0$;
- $G_2(s)$ is the transfer function of a non-minimum phase system, it has thus an inverse response;
- $G_3(s)$ has one zero in the left half plane that is on the left of the two poles, the step response cannot have overshoot.

$G_4(s)$ is the only transfer function whose step response can be characterised by overshoot and $\dot{y}(0) \neq 0$.