Automatic Control

Motion control
Advanced control techniques

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Besides the classical P/PI motion control architecture, there are other control architectures that can integrate or substitute the classical one.

Some of these architectures are part of commercial motion control systems.

We will concentrate on the following topics:

- notch filter
- torque disturbance observer
- state space design
- input shaping

A notch filter is a kind of band-stop filter whose aim is to stop a single frequency or, equivalently, to substitute a pair of complex conjugate poles, usually characterized by low damping, with another one characterized by higher damping.

A notch filter is characterized by the following transfer function

$$G_{nf}(s) = \frac{s^2 + 2\xi_1 \omega_n s + \omega_n^2}{s^2 + 2\xi_2 \omega_n s + \omega_n^2}$$

where ω_n is the natural frequency of the poles we are cancelling out and ξ_1, ξ_2, with $\xi_2 > \xi_1$, are the damping coefficients.

The filter can be also specified giving the natural frequency and the -3dB bandwidth.
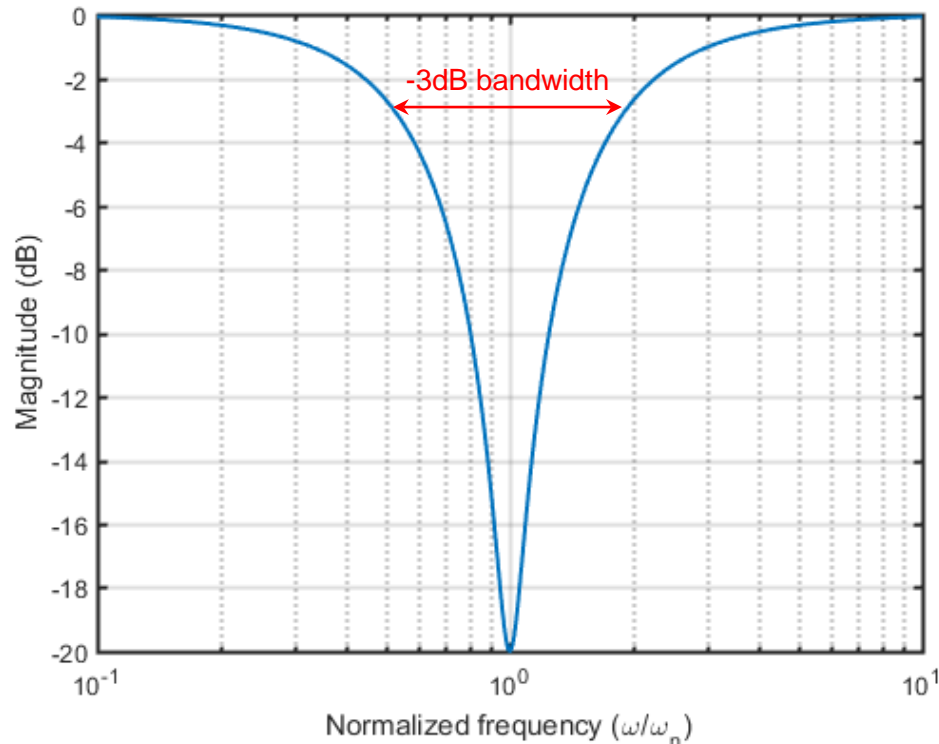
In motion control systems, a notch filter can be used in the velocity loop to filter the output of the PI controller.
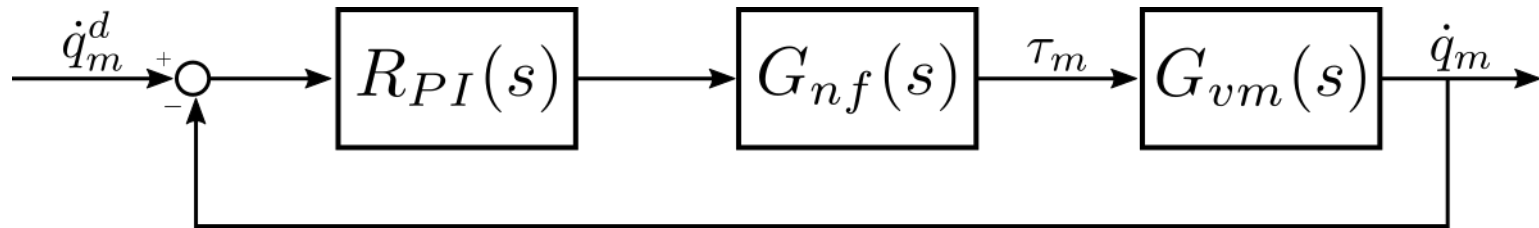
The notch filter can be used to cancel out an high frequency vibration (out of the control bandwidth) that causes audible noise.

We could also study if using the notch filter to cancel out the first natural frequency of the motion control system (joint flexibility) allows to increase control performance.

In this particular case the notch filter will have the following expression

$$G_{nf}(s) = \frac{s^2 + 2\hat{\xi}_p \hat{\omega}_p s + \hat{\omega}_p^2}{s^2 + 2\xi_2 \hat{\omega}_p s + \hat{\omega}_p^2}$$

where $\hat{\omega}_p$ and $\hat{\xi}_p$ are an estimate of the natural frequency and damping of the first resonance.

Using the filter to cancel out the first resonance frequency has, however, some drawbacks:

- in order to cancel out the first resonance frequency, an accurate estimate of the natural frequency of the pair of complex and conjugate poles is required

- the pair of low damping poles cancelled out by the notch filter (from the reference-output transfer function) are still eigenvalues of the closed-loop system and appear as poles in other transfer functions, e.g., the disturbance-output transfer function

- transforming an analog into a digital filter causes a distortion in the frequency response that could slightly change the frequency of the zeros in the digital realization. To overcome this issue we can use frequency pre-warping
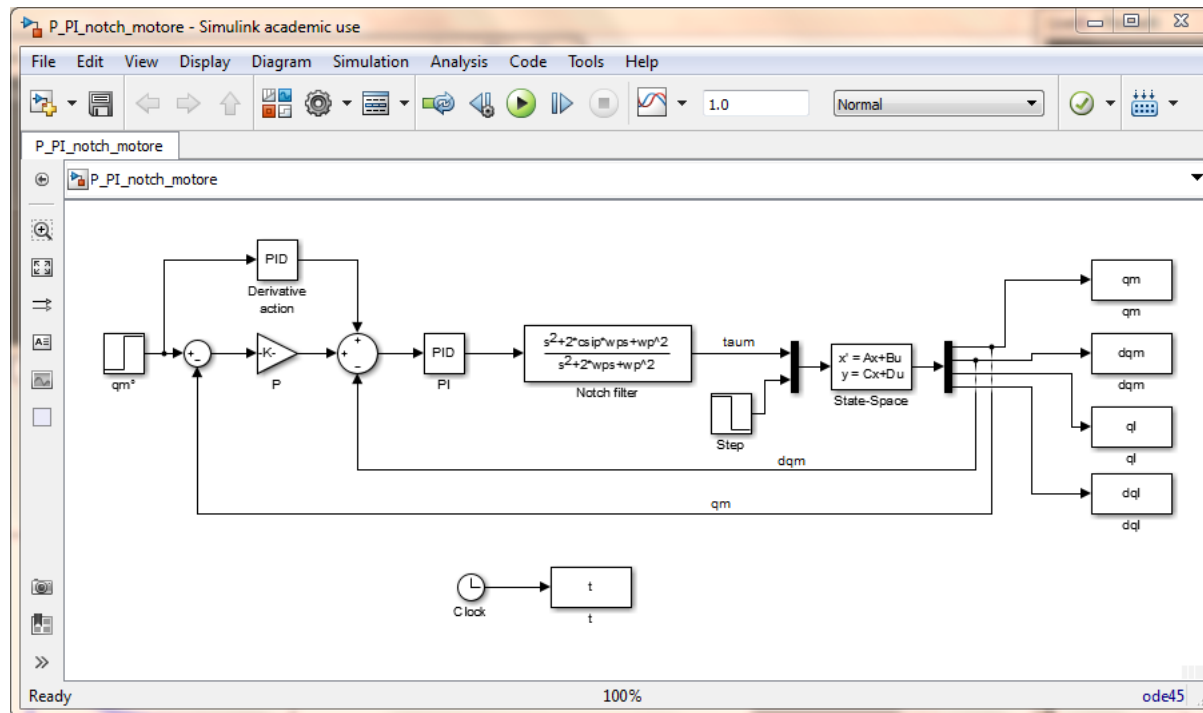
Let's see the results of the P/PI control, together with a notch filter in the velocity loop, on a simulation example.
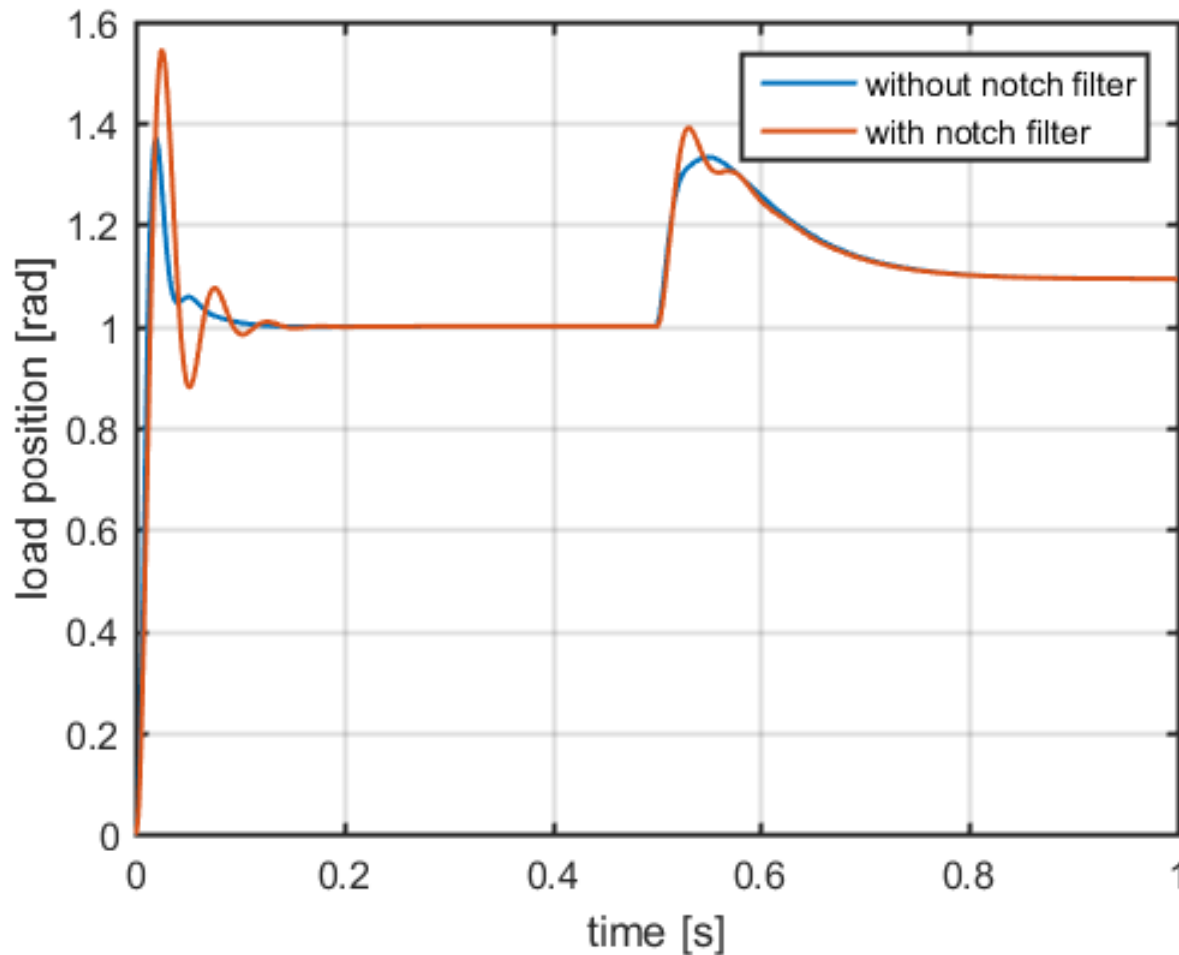
The main characteristics of the servomechanism are

$$\omega_z = 200 \; rad/s \quad \rho = 1 \quad \xi_z = 0.1$$

For the P/PI control

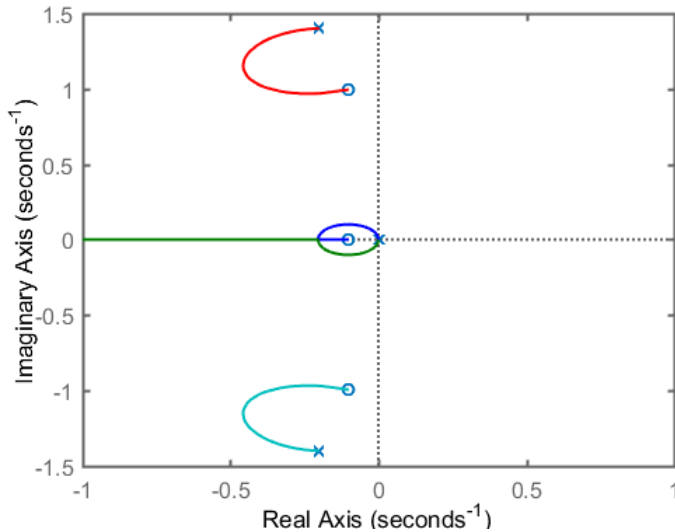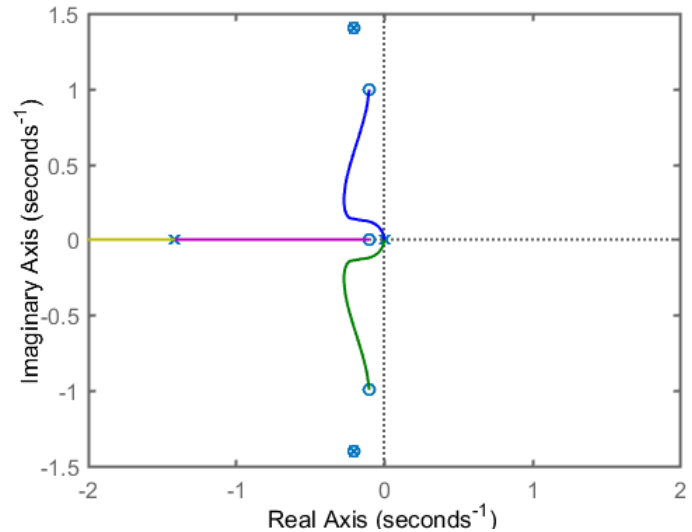$$T_{I_v} = 10/\omega_z \quad \tilde{\omega}_{c_v} = 1 \quad \tilde{\omega}_{c_p} = 0.1$$

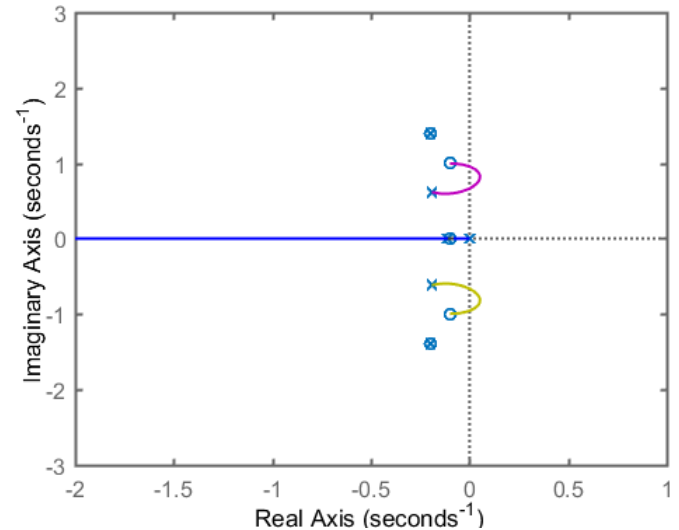The performance obtained with the notch filter is worse than the one achieved without the filter.

Notch filter – Simulation example (III)
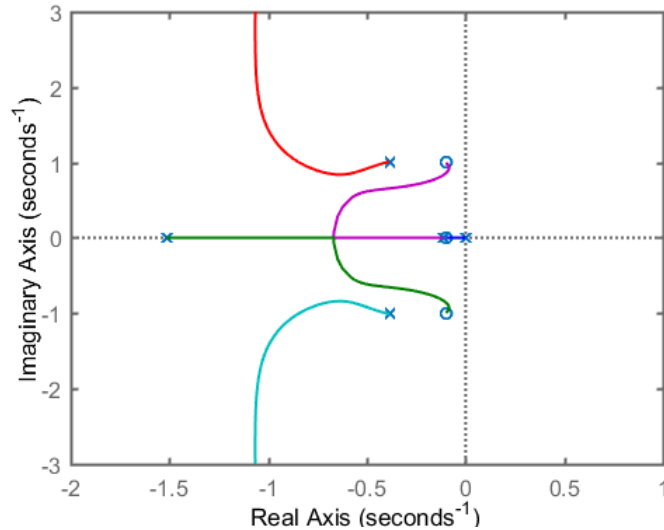
Without filter

With filter

Velocity loop

Position loop

Damping of the closed-loop poles is not increased by the notch filter.

We can adopt a different strategy, that should allow to improve the damping of the closed-loop poles.
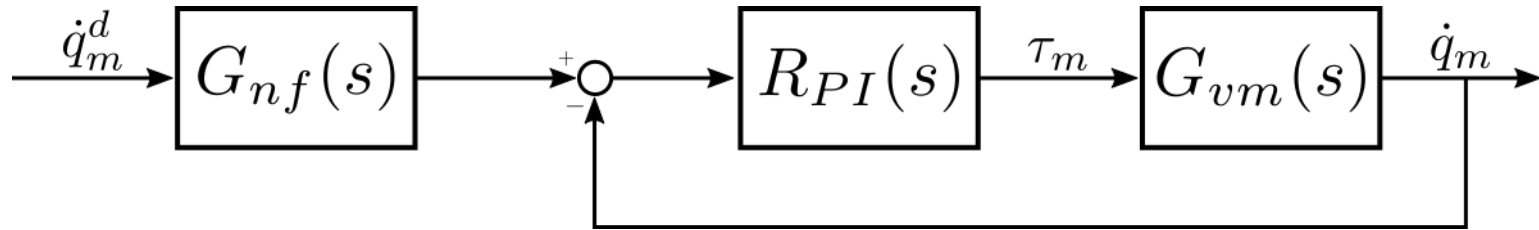
In this case, the zeros of the notch filter are selected to match the pair of complex poles of the closed-loop system.

This strategy can achieve good performance, keeping the same tuning of the velocity regulator one would use without the filter.

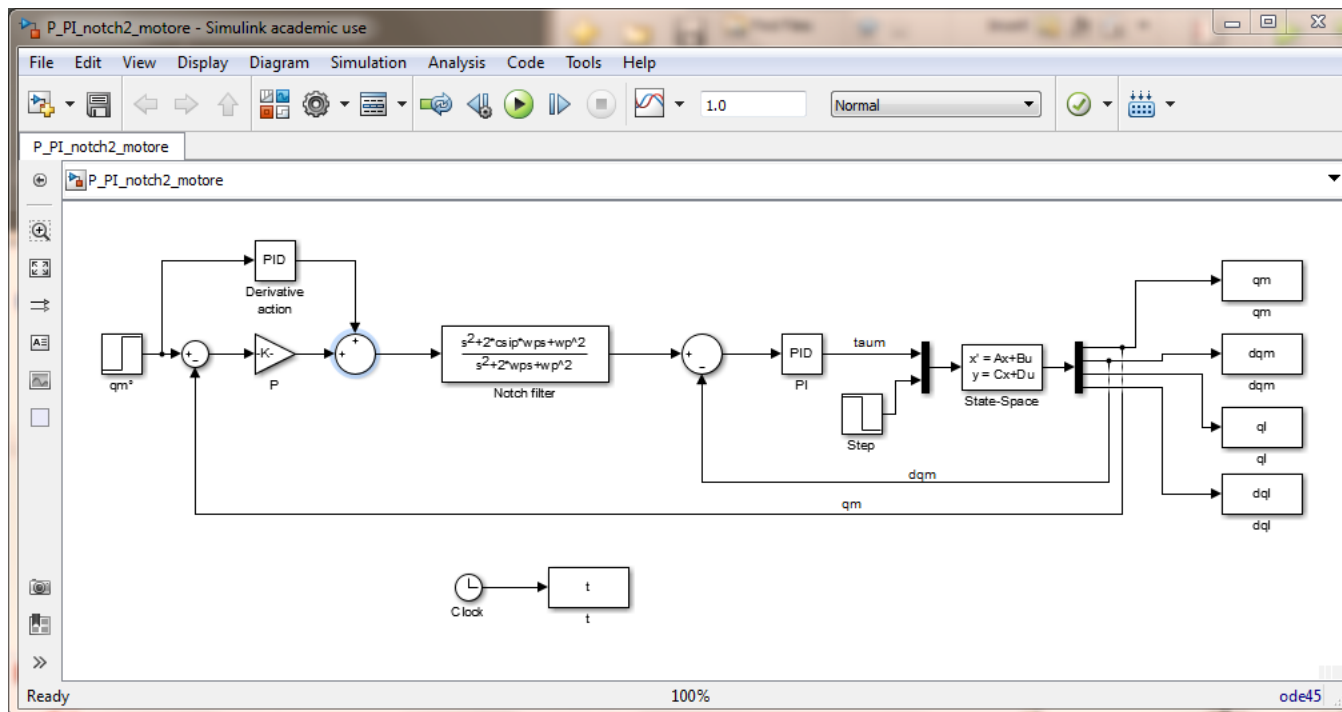Let's see the results of the P/PI control, together with a notch filter on the velocity reference, on a simulation example.
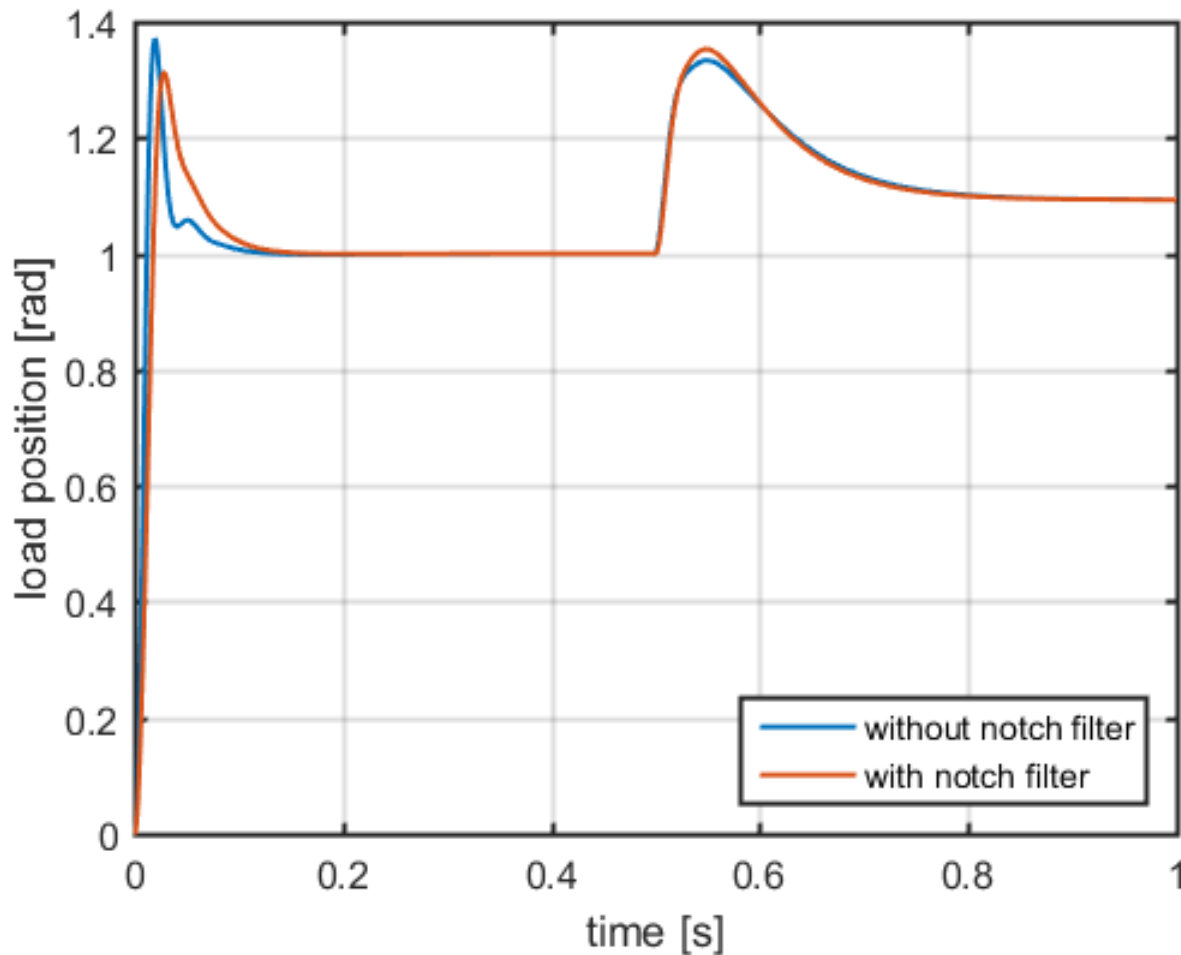
The main characteristics of the servomechanism are

$$\omega_z = 200 \; rad/s \quad \rho = 1 \quad \xi_z = 0.1$$

For the P/PI control

$$T_{I_v} = 10/\omega_z \quad \tilde{\omega}_{c_v} = 1 \quad \tilde{\omega}_{c_p} = 0.1$$

There is a significant improvement in the load side response.

In many motion control applications there is a significant load torque disturbance, whose compensation can greatly improve the performance of the control system.

Torque Disturbance Observer (TDO) is an architecture that, acting in parallel to the motion control system, estimates and compensates the disturbance.

We start studying the problem of estimating and compensating a load disturbance acting on a general plant.

$$u \xrightarrow{\ +\ } \bigcirc \xleftarrow[\ ]{\ d\ } \xrightarrow{\ +\ } \boxed{P(s)} \xrightarrow{\ y\ }$$
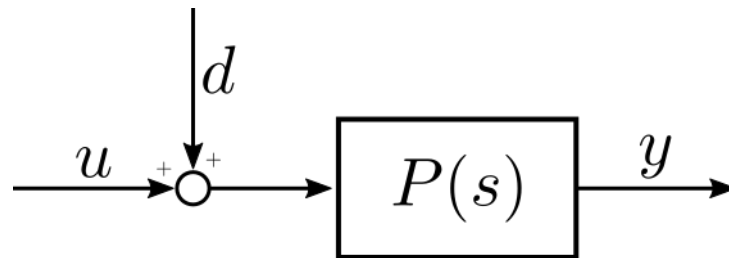
Let's consider the following architecture, where

- $P_n(s)$ is a model of the plant $P(s)$
- $Q(s)$ is a unitary gain low-pass filter that makes causal the transfer function $Q(s)P_n^{-1}(s)$
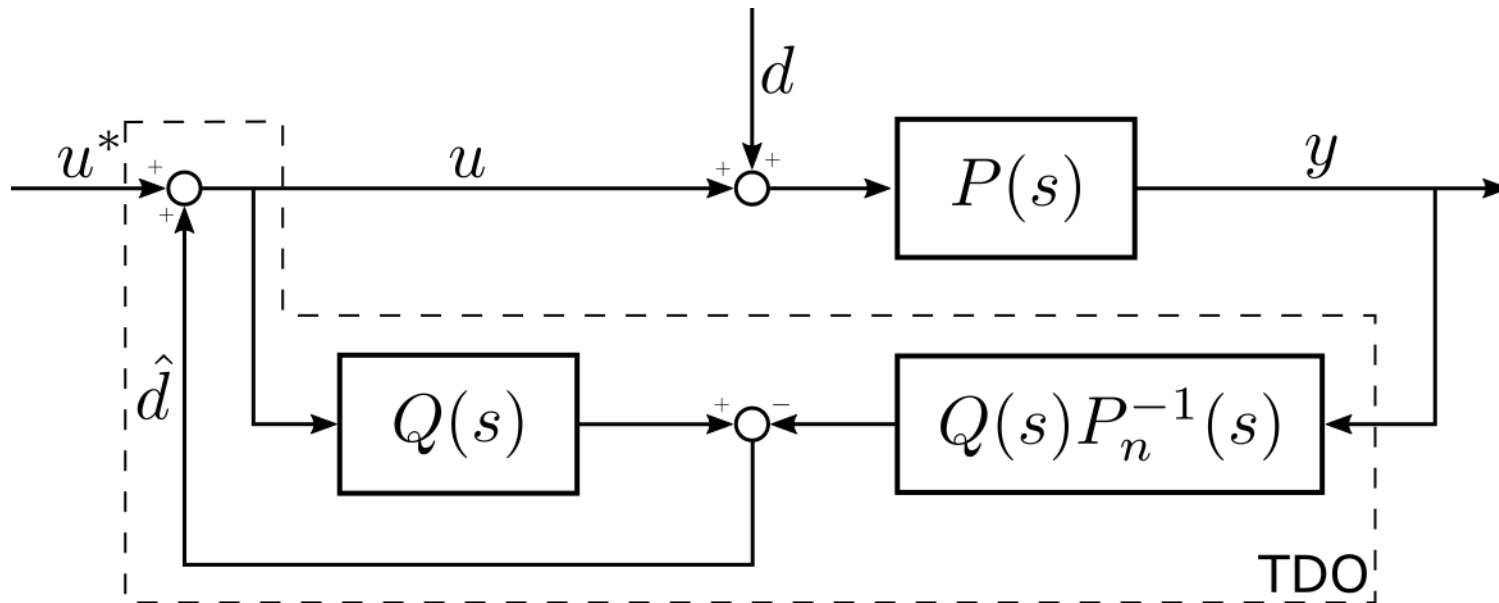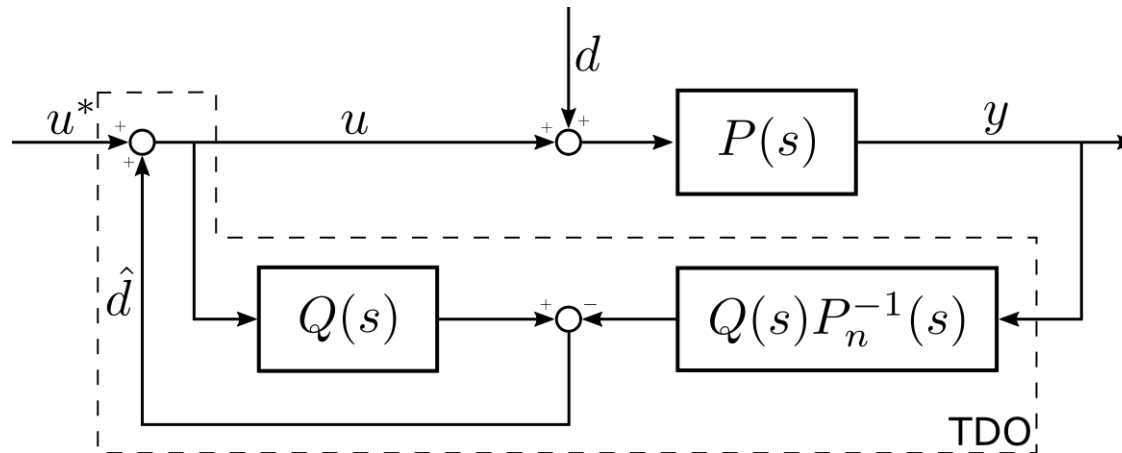
TDO produces the following disturbance estimate

$$\hat{D}(s) = Q(s)U(s) - P_n^{-1}(s)Q(s)Y(s)$$

$$= Q(s)U(s) - P_n^{-1}(s)Q(s)P(s)\left(D(s)+U(s)\right)$$

$$= Q(s)\left(1 - P_n^{-1}(s)P(s)\right)U(s) - Q(s)P_n^{-1}(s)P(s)D(s)$$

and if $P_n(s) \approx P(s)$ we get

$$\hat{D}(s) \approx -Q(s)D(s)$$

the harmonics of the disturbance inside the bandwidth of the filter are correctly estimated.
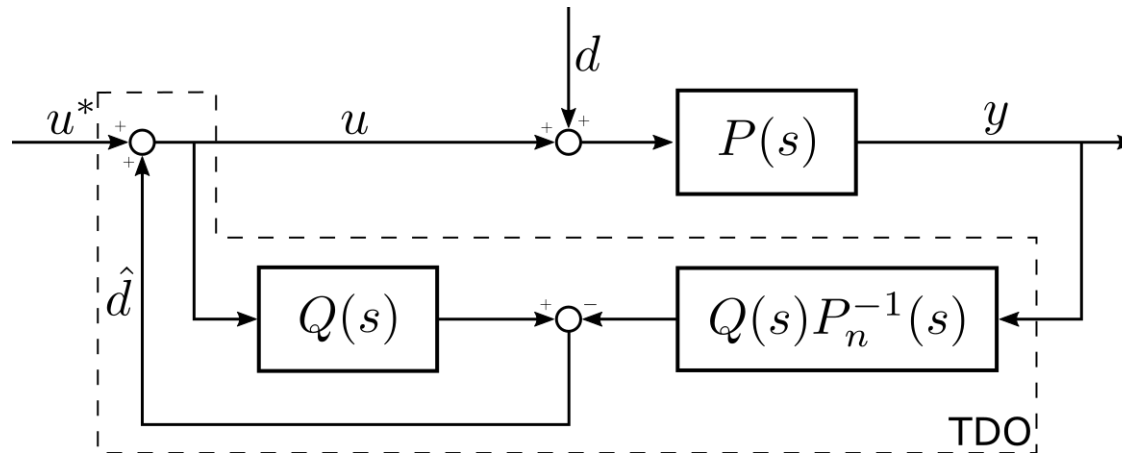
Let's consider now the transfer function from u^* to y

$$Y(s) = \frac{P(s)\left(1-Q(s)\right)}{1-Q(s)+P_n^{-1}(s)P(s)Q(s)}D(s) + \frac{P(s)}{1-Q(s)+P_n^{-1}(s)P(s)Q(s)}U^*(s)$$

if $P_n(s) \approx P(s)$ we get

$$Y(s) = P(s)\left(1-Q(s)\right)D(s) + P(s)U^*(s)$$

We can thus conclude that the system (from u^* to y) is virtually not affected by the disturbance, at least for the harmonics whose frequency lies in the bandwidth of $Q(s)$.

Consider now a servomechanism, and in particular the torque-velocity transfer function

$$P(s) = \frac{1}{J_m s + D_m}$$

and select for $Q(s)$ a first order low-pass filter

$$Q(s) = \frac{1}{1 + sT_f}$$

The estimate of the torque disturbance is given by

$$\hat{\tau}_d(s) = Q(s)\tau_m(s) - Q(s)P_n^{-1}(s)\dot{q}_m(s) = \frac{1}{1 + sT_f}\tau_m(s) - \frac{J_m s + D_m}{1 + sT_f}\dot{q}_m(s)$$

this relation can be rewritten as

$$\hat{\tau}_d(s) = \frac{1}{1 + sT_f}\left[\tau_m(s) - \left(D_m - \frac{J_m}{T_f}\right)\dot{q}_m(s)\right] - \frac{J_m}{T_f}\dot{q}_m(s)$$

The previous relation is used to devise the standard TDO architecture.

$$\hat{\tau}_d(s) = \frac{1}{1 + sT_f}\left[\tau_m(s) - \left(D_m - \frac{J_m}{T_f}\right)\dot{q}_m(s)\right] - \frac{J_m}{T_f}\dot{q}_m(s)$$

Disturbances

Computing the transfer function from the disturbances and the reference signal to the output we get

$$q_m(s) = \frac{1}{J_m s^2 + D_m s} \left[\tau_m^*(s) - G_f(s)\left(\tau_c(s) + \tau_{ms}(s) \right) \right]$$
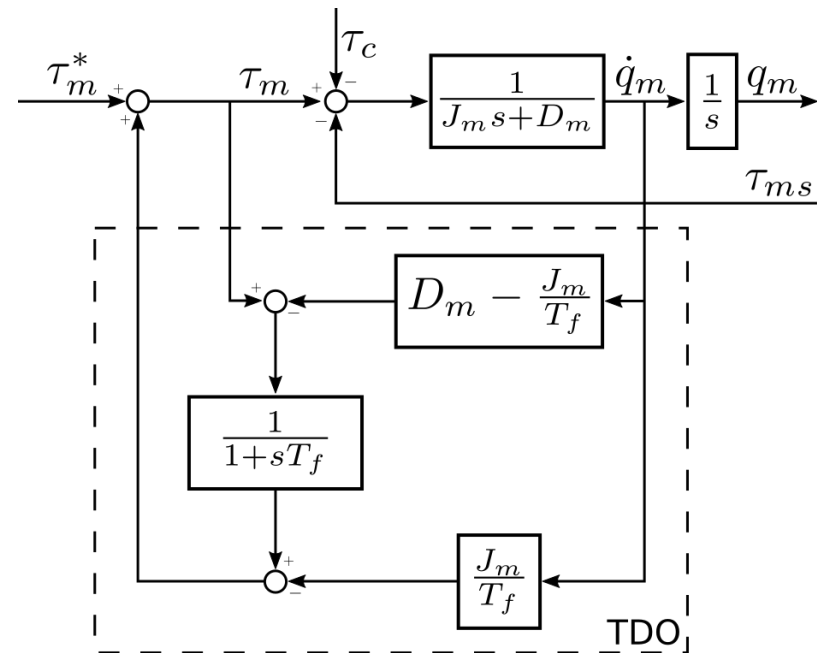
where

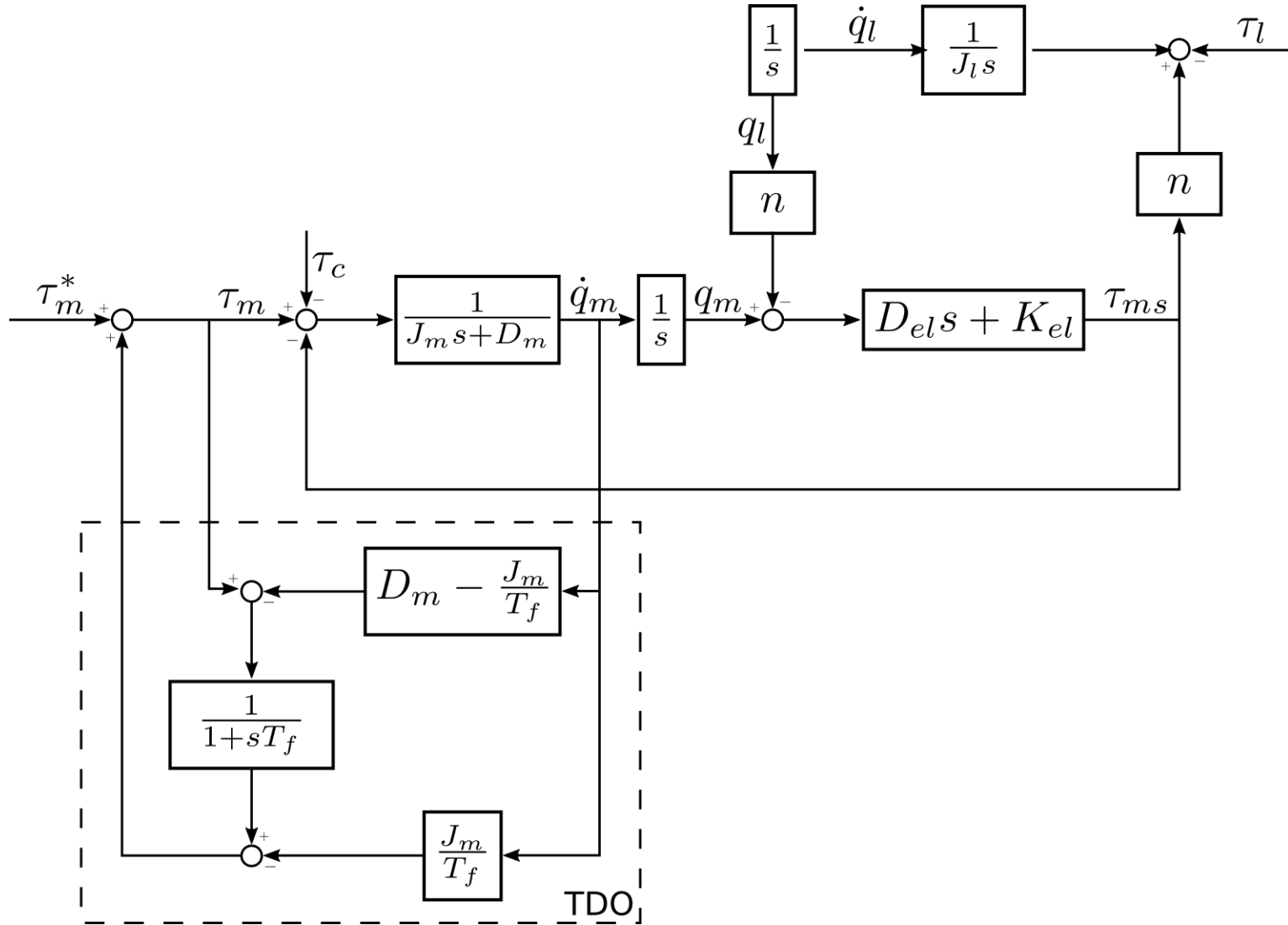$$G_f(s) = \frac{sT_f}{1 + sT_f}$$

is a high-pass filter.

From the transfer function it is evident that the disturbance is very well filtered, at least at low frequencies.

Remember that the TDO has been designed considering the rigid model of the servomechanism.

What happens if we consider the elastic model?

Let's compute the system transfer functions considering $\tau_l = 0$

$$q_m(s) = \frac{J_{lr}s^2 + D_{el}s + K_{el}}{H_f(s)} \left[\tau_m^*(s) - G_f(s)\tau_c(s) \right]$$

$$q_l(s) = \frac{D_{el}s + K_{el}}{H_f(s)} \left[\tau_m^*(s) - G_f(s)\tau_c(s) \right]$$

where

$$H_f(s) = \left(J_{lr}s^2 + D_{el}s + K_{el} \right) \left(J_m s^2 + D_m s \right) + G_f(s) \left(D_{el}s + K_{el} \right) J_m s^2$$

When $T_f \to 0$ we get

$$q_m(s) \approx \frac{1}{J_m s^2 + D_m s} \tau_m^*(s)$$

$$q_l(s) \approx \frac{D_{el}s + K_{el}}{J_{lr}s^2 + D_{el}s + K_{el}} q_m(s)$$

- The torque disturbance has been completely rejected
- The motor is controlled as it was independent of the load
- The load oscillates at the locked-frequency
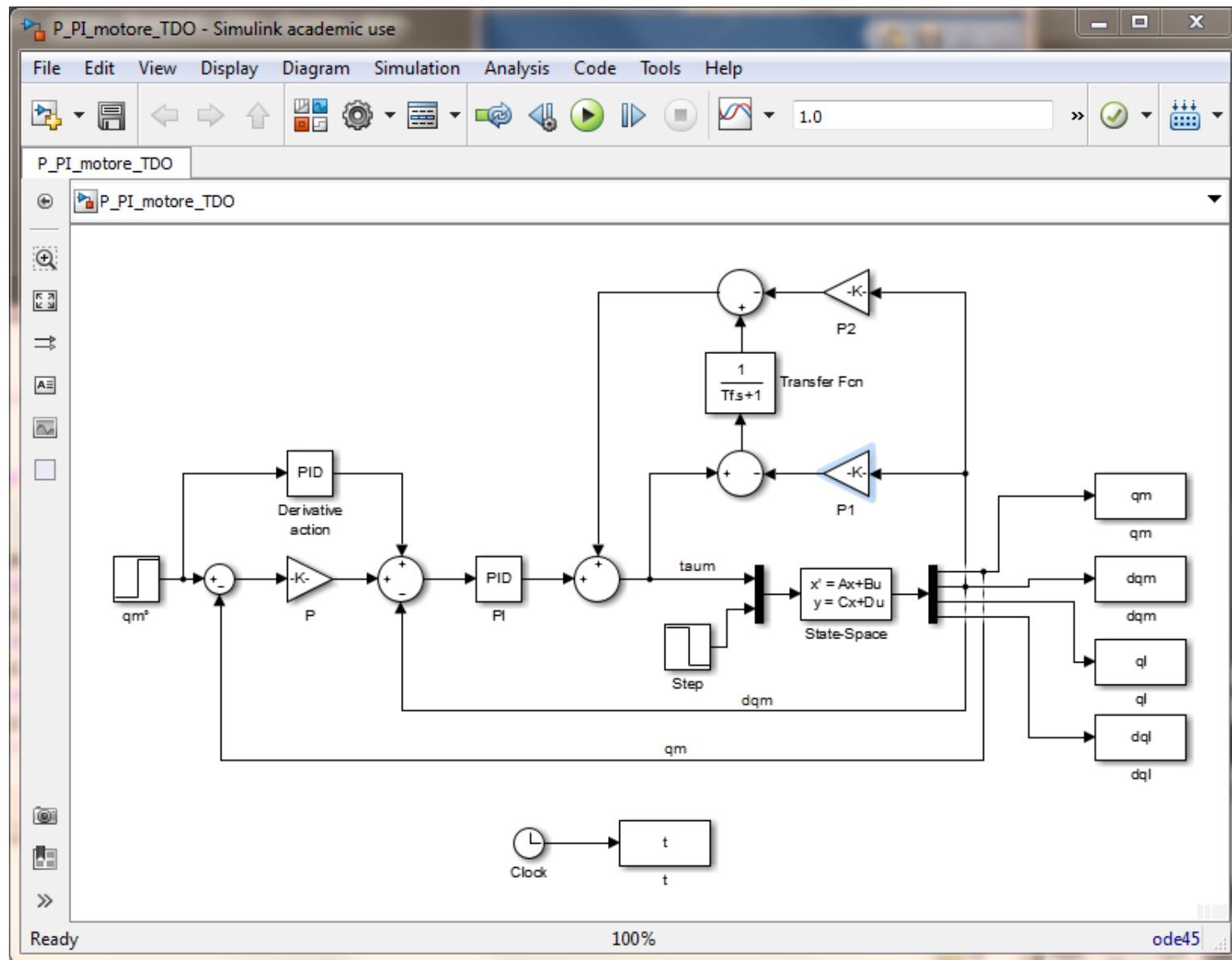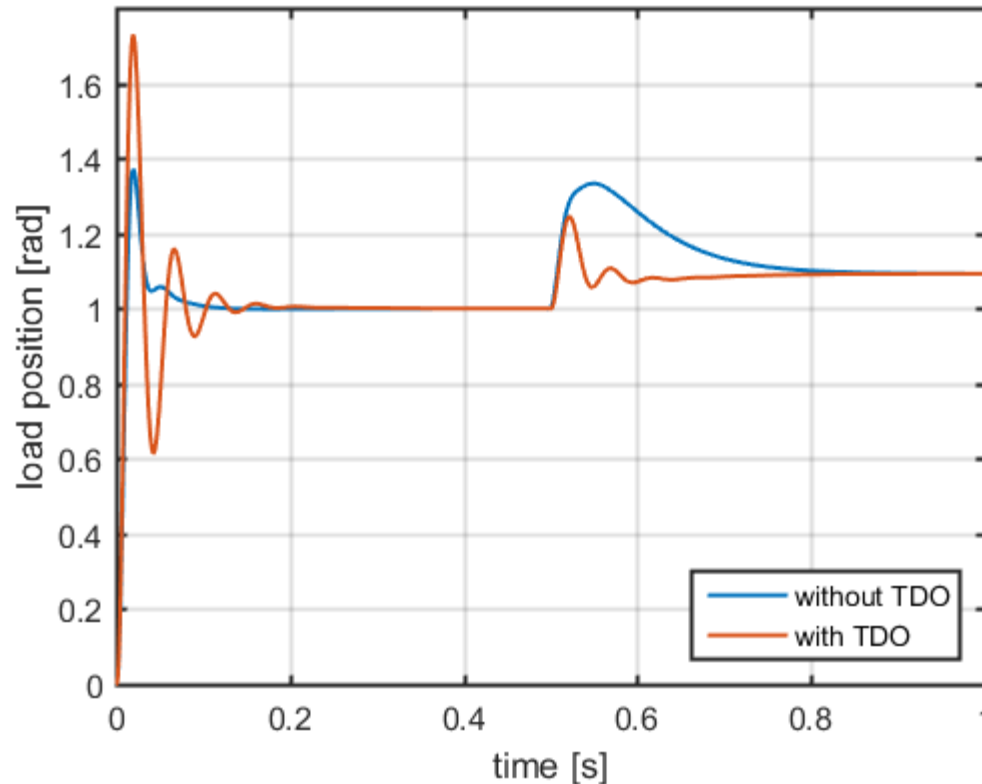
The Torque Disturbance Observer interprets the load torque generated by the transmission as a disturbance to be rejected.

In the case of elastic transmissions, TDO does not achieve good performance.

The techniques we have studied show that even controlling with the best possible performance the motor does not imply that load side control performance is acceptable.

Instead, there are situations in which we perfectly control the motor position but the load position is not controlled.

Pole placement should allow to take into account all the system states, thus ensuring good performance with respect to the motor and the load as well.

Let's first introduce a state-space representation of the model of an elastic servomechanism

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix} q_m & \dot{q}_m & nq_l & n\dot{q}_l \end{bmatrix}^T \qquad u = \tau_m \qquad y = q_m$$

The three state-space matrices are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_{el}}{J_m} & -\dfrac{D_m+D_{el}}{J_m} & \dfrac{K_{el}}{J_m} & \dfrac{D_{el}}{J_m} \\ 0 & 0 & 0 & 1 \\ \dfrac{K_{el}}{J_{lr}} & \dfrac{D_{el}}{J_{lr}} & -\dfrac{K_{el}}{J_{lr}} & \dfrac{D_{el}}{J_{lr}} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{J_m} \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

where $J_{lr} = J_l/n^2$.

It can be easily checked that the system is completely controllable and completely observable.

Our aim is to design an observer based pole placement control law with reference tracking and feedforward action.
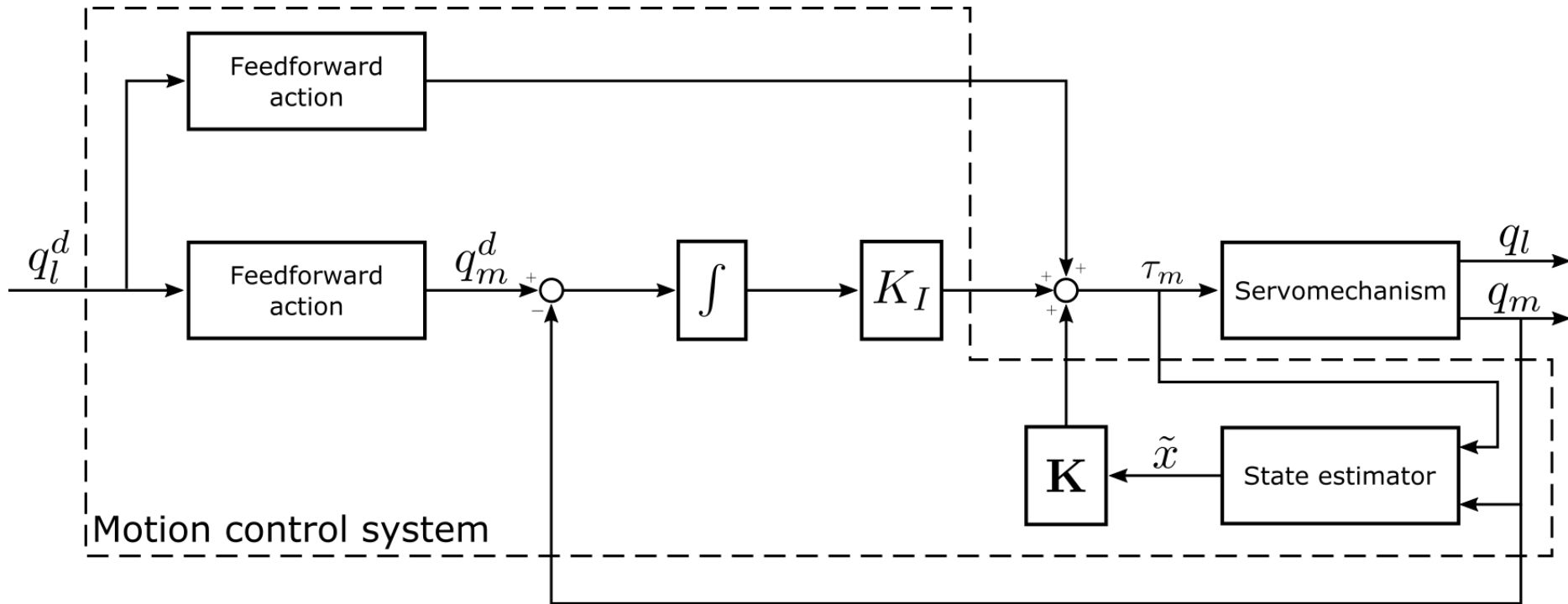
The integral action allows to achieve zero steady-state error in the presence of a constant reference and of constant disturbances (e.g., Coulomb friction).

Introducing the integrator state x_I we get

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\dot{x}_I = \bar{y} - y = -\mathbf{C}\mathbf{x} + \bar{y}$$

We can define a new state vector for the augmented system

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ x_I \end{bmatrix}$$

and rewrite the system as

$$\dot{\mathbf{z}} = \mathbf{F}\mathbf{z} + \mathbf{G}_u\mathbf{u} + \mathbf{G}_y\bar{y}$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \quad \mathbf{G}_u = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \quad \mathbf{G}_y = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

A few remarks:

- thanks to the separation principle we can design the pole placement control law assuming full-state feedback, and solving separately the observer design problem

- in order to place the poles of the augmented system, $(\mathbf{F}, \mathbf{G}_u)$ has to be completely controllable, i.e., $(\mathbf{A}, \mathbf{B})$ is completely controllable and the transfer function of the controlled system has no zeroes in $s = 0$

As the last condition is verified, we can arbitrarily assign the closed-loop poles of the augmented system, as follows

$$\dot{\mathbf{z}} = \mathbf{F}\mathbf{z} + \mathbf{G}_u\mathbf{u}$$

$$u = \begin{bmatrix} \mathbf{K} & k_I \end{bmatrix} \mathbf{z}$$

and

$$\mathbf{K}_{tot} = \begin{bmatrix} \mathbf{K} & k_I \end{bmatrix} \quad \Rightarrow \quad \dot{\mathbf{z}} = \left(\mathbf{F} + \mathbf{G}_u\mathbf{K}_{tot} \right) \mathbf{z}$$

A few remarks:

- ideally closed-loop poles can be arbitrarily selected, in practice robustness of the pole placement control law is closely related to the desired positions of the closed-loop poles

- as a measure of robustness we can take the condition number of the closed-loop system eigenvector matrix (Bauer-Fike theorem): the more orthogonal the eigenvectors are, the better the condition number is, and the more robust the closed-loop system is

Another way to select the gains of the control law is by minimizing the following cost function

$$J = \int_0^\infty \left[\mathbf{z}(t)^T \mathbf{Q} \mathbf{z}(t) + u(t)^2 \right] dt \qquad \mathbf{Q} \geq 0$$

that entails the solution of a Linear Quadratic optimal control problem.

You can try to follow this alternative way, computing matrix $\mathbf{K}_{tot}$ by way of the Matlab function lqr.

How to select matrix $\mathbf{Q}$?

You can use a trial-and-error procedure, or make reference to LQ optimal control literature.

Let's now consider the state observer, whose equations are

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{L}\left(\tilde{y} - y\right)$$

$$\tilde{y} = \mathbf{C}\tilde{\mathbf{x}}$$

The dynamics of the error system are instead described by

$$\dot{\varepsilon} = \left(\mathbf{A} + \mathbf{L}\mathbf{C}\right)\varepsilon$$

A few remarks:

- the observer can be designed if $(\mathbf{A}, \mathbf{C})$ is completely observable (in this case the system is completely observable)

- the higher the absolute value of the eigenvalues of $\mathbf{A} + \mathbf{L}\mathbf{C}$ is, the faster the error converges to zero, but the more sensitive the estimate is to measurement noise

- observer design, as pole placement, can be alternatively reformulated as the minimization of an integral cost function, using <u>Kalman filter theory</u> (in this case, however, the model of the system should be stochastic, and disturbances become stochastic processes)

To improve the performance of the closed-loop system, we can introduce a feedforward action.
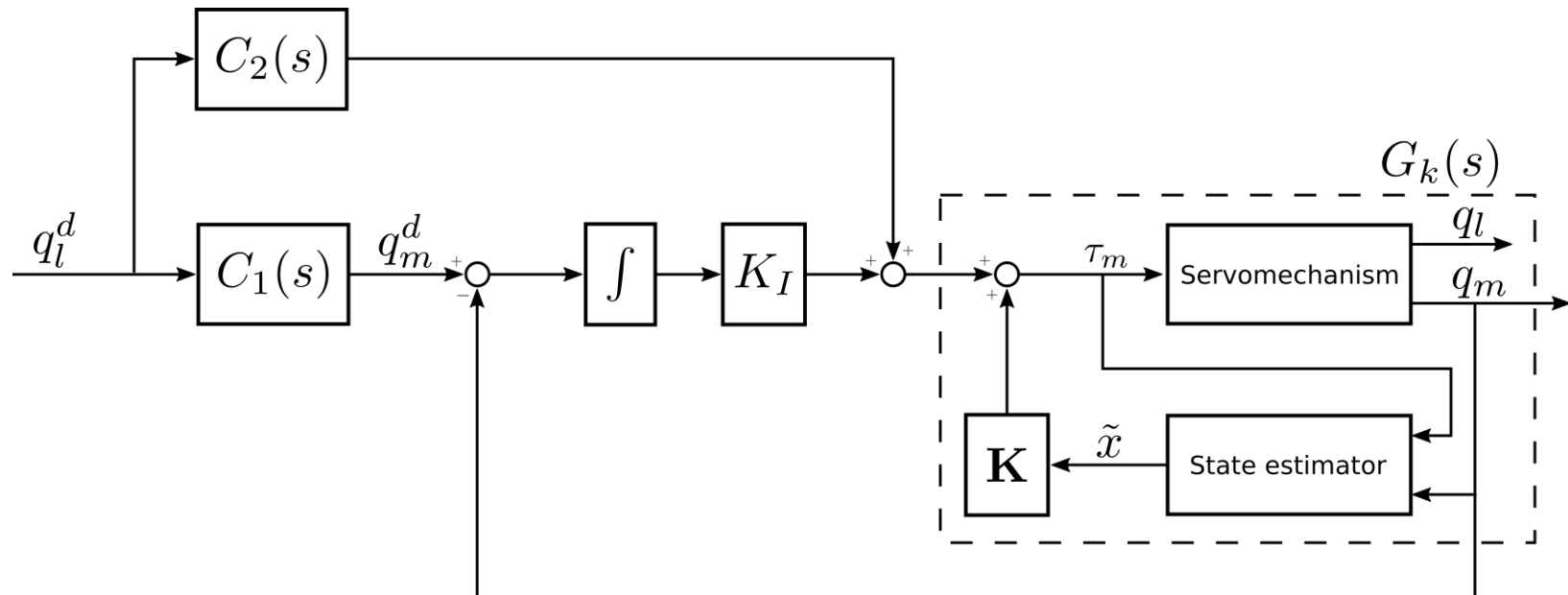
The transfer function of the system in the box is given by

$$G_k(s) = \mathbf{C}\left(s\mathbf{I}_4 - (\mathbf{A} + \mathbf{BK})\right)^{-1}\mathbf{B} = \frac{B_m(s)}{\chi_{\mathbf{A}+\mathbf{BK}}(s)}$$

where $B_m(s)$ is the numerator of the transfer function from τ_m to q_m.

Define the following transfer function

$$G_{lm}(s) = \frac{B_l(s)}{B_m(s)}$$

where $B_l(s)$ ($B_m(s)$) is the numerator of the transfer function from τ_m to q_l (q_m).

Assuming as reference model

$$\frac{q_l(s)}{q_l^d(s)} = F^o(s)$$

the feedforward compensators can be selected as

$$C_1(s) = G_{lm}^{-1}(s)F^o(s) \qquad C_2(s) = G_k^{-1}(s)G_{lm}^{-1}(s)F^o(s)$$

In order to have causal filters, the relative degree of the reference model should be greater or equal to 3.

Torque constant K_t	1.6 Nm/A_{rms}
Motor inertia J_m	0.00015 Kgm^2
Load inertia J_l	2.7 Kgm^2
Transmission ratio n	100
Viscous friction D_m	0.0034 Nms/rad
Stiffness constant K_{el}	3.1 Nm/rad
Elasticity damping coeff. D_{el}	0.0022 Nms/rad
Antiresonance frequency ω_{tz}	105 rad/s
Complex zeros damping ζ_z	0.063
Resonance frequency ω_{tp}	179 rad/s
Complex poles damping ζ_p	0.138
Real pole frequency $1/T$	9.12 rad/s

Experiment	1	2	3
Load rotation (deg)	40	30	50
Time for the positioning (s)	0.5	0.5	0.9
Maximum acceleration (rad/s^2)	1200	900	500
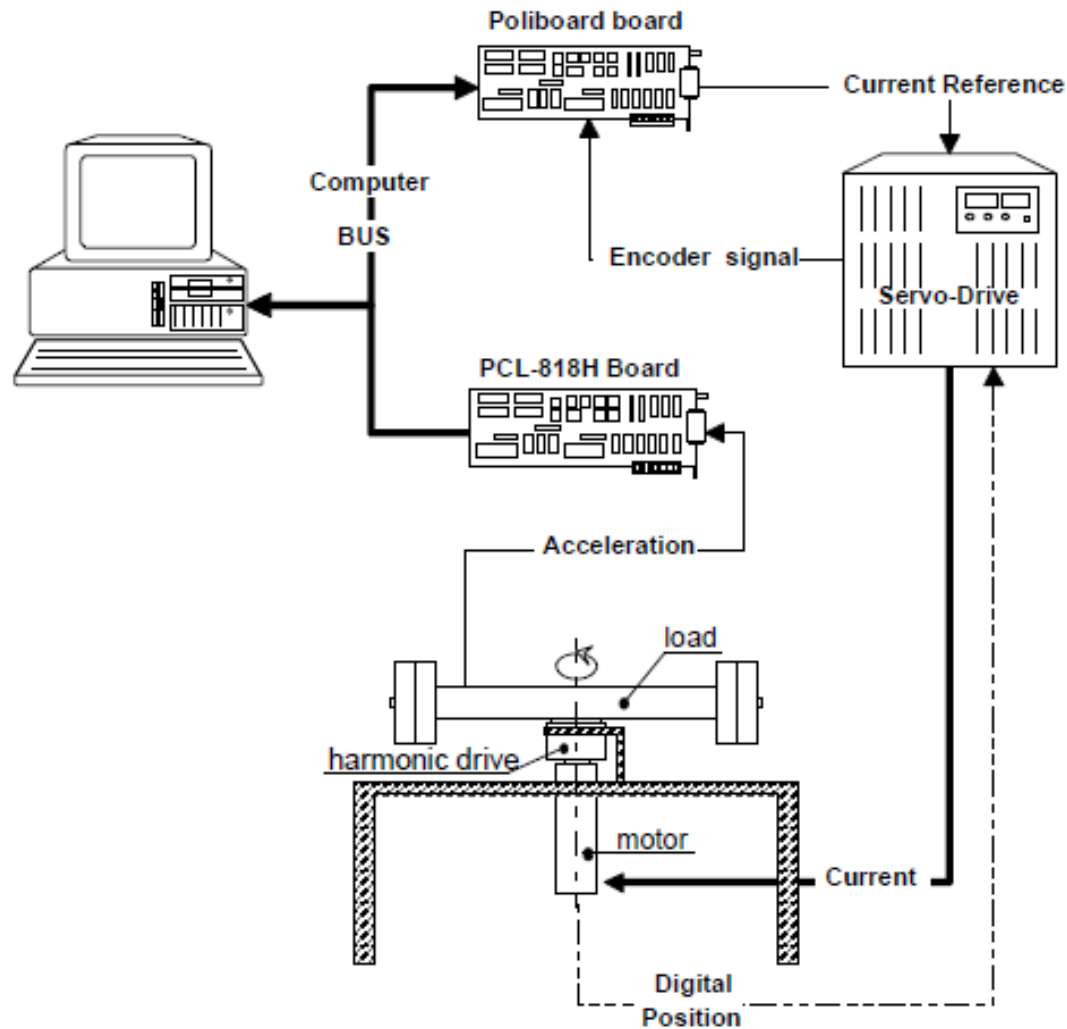
We would like to consider the following set-up:

- a servomechanism constituted by a brushless motor, an Harmonic Drive transmission, an inertial load

- a sensory system constituted by a motor side encoder and a load side accelerometer (from which load velocity is computed for validation purpose)

The aim of the experiment is to compare the performance of two different motion control systems:

- PID control (P/PI loop with velocity feedforward)

- LQG control (LQ optimal control and Kalman filter)

Performance is compared using a trapezoidal velocity profile at different velocity/acceleration values, comparing the load velocity behavior.
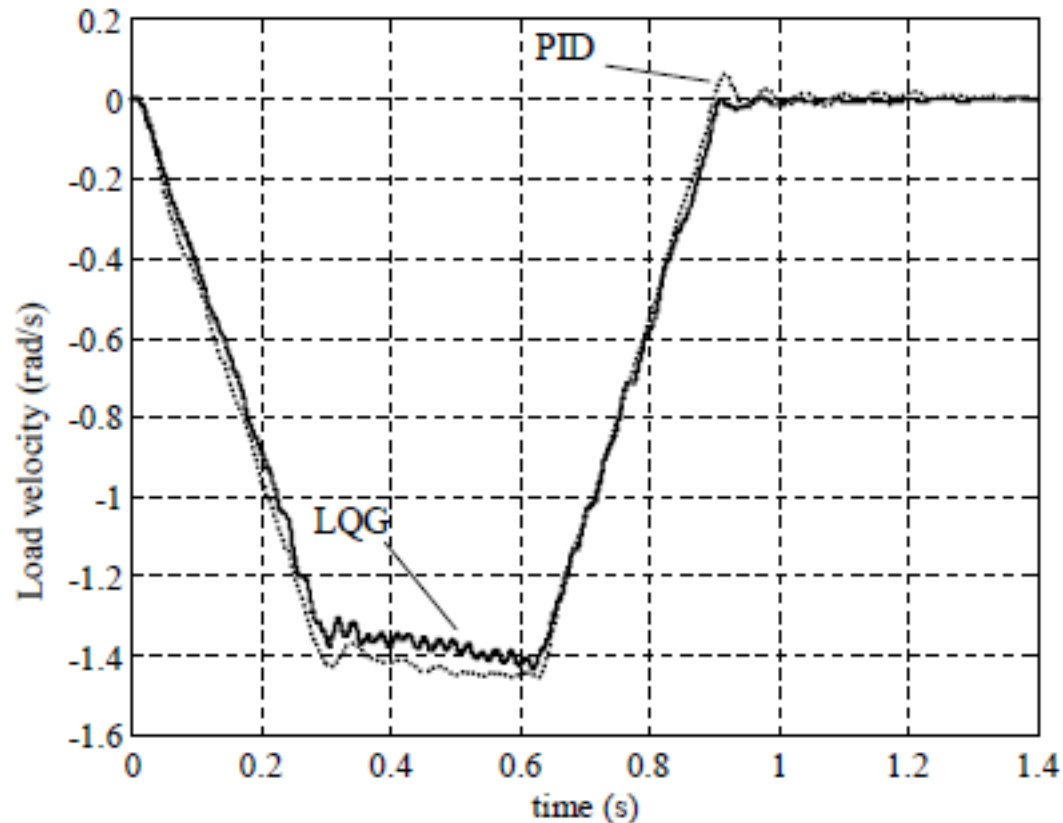
Performing a rotation of 40° load side in 0.5 seconds with a maximum acceleration of 1200 rad/s² (motor side).
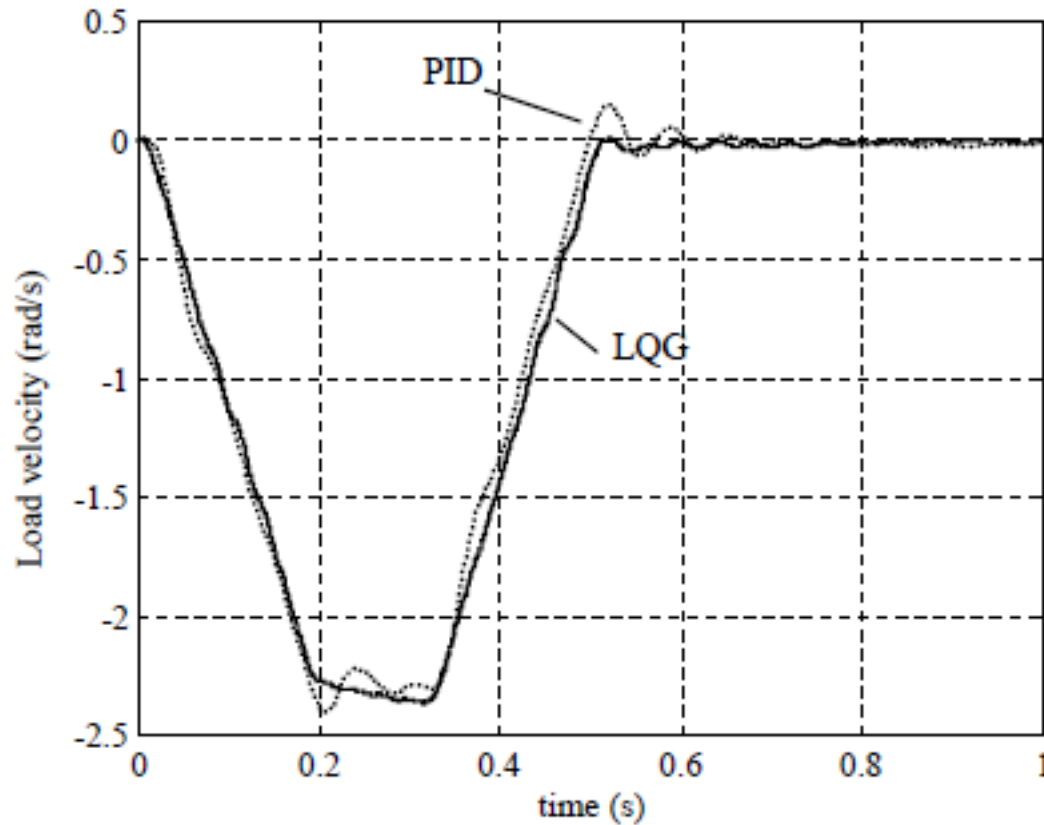
Performing a rotation of 30° load side in 0.5 seconds with a maximum acceleration of 900 rad/s² (motor side).

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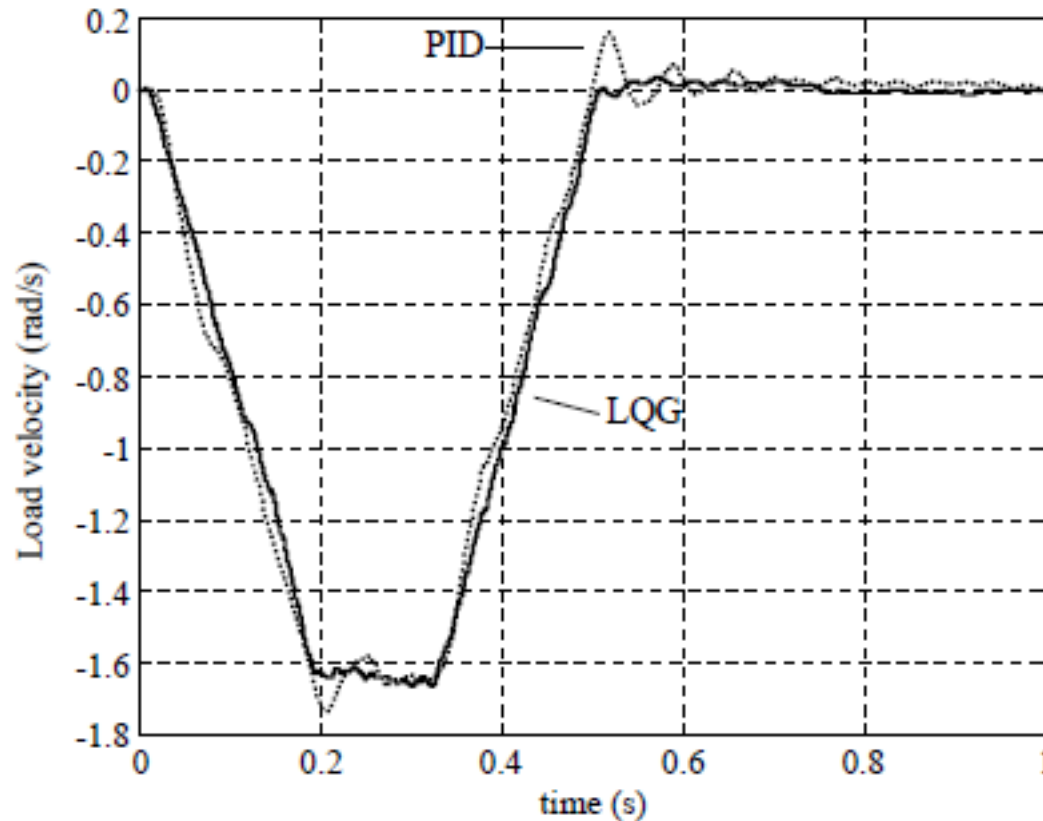
Performing a rotation of 50° load side in 0.9 seconds with a maximum acceleration of 500 rad/s² (motor side).

At low velocity a ripple arises that is probably due to disturbances in the velocity estimation.

We conclude that LQG gives rise to better results than PID control, this better performance, however, entails a more complex design, implementation and debugging of the control system.

There are many examples of robots that are affected not only by joint elasticity, but even by link flexibility.

This is not just the case of space robotic arms, even industrial arms can exhibit link flexibility.

To cope with either link or joint flexibility, we will introduce a feedforward technique called input shaping.

This technique modifies the input to the system in such a way that the effect of the mechanical resonances is canceled out.
In order to apply this methodology we need to know the natural frequency and damping of the pair of complex and conjugate poles.
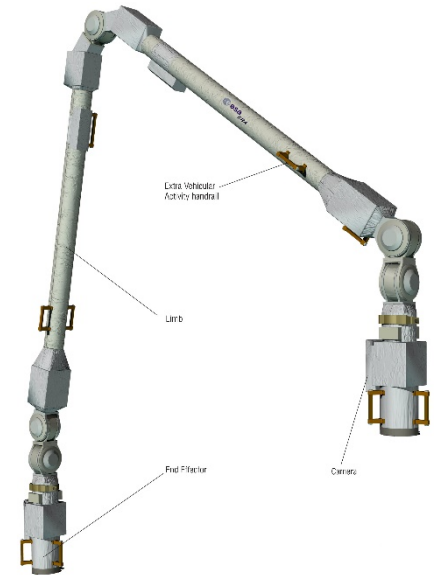
Let's consider a dynamical system characterized by a pair of complex poles

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

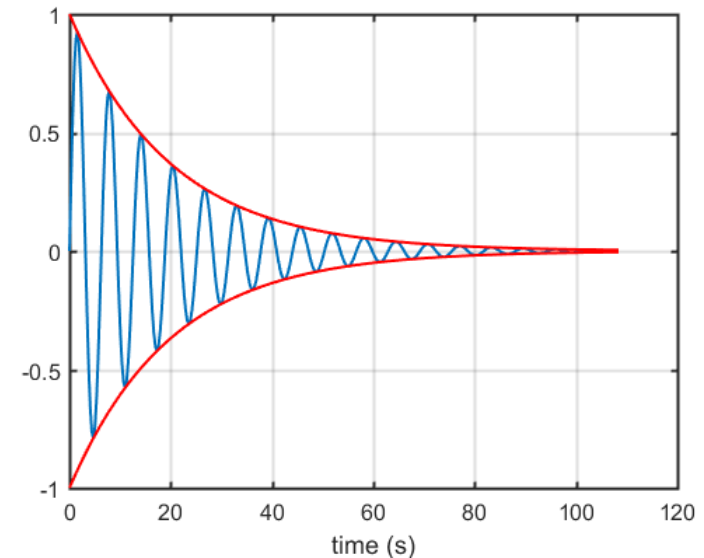The response of the system to an impulse of amplitude k_i at time t_i is given by

$$y_i(t) = B_i e^{-\xi\omega_n(t-t_i)} \sin\left(\omega_n\sqrt{1-\xi^2}(t-t_i)\right) \quad t \geq t_i$$

where

$$B_i = k_i \frac{\omega_n}{\sqrt{1-\xi^2}}$$

and the period of oscillations is given by

$$\Delta T = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$$

Let's consider now the response of the system to <u>two impulses</u>, at t_1 and t_2 ($t_2 > t_1$). The response of the system is the sum of the two impulse responses

$$y(t) = B_1 e^{-\xi \omega_n (t-t_1)} \sin \left(\omega_n \sqrt{1 - \xi^2} (t - t_1) \right) + B_2 e^{-\xi \omega_n (t-t_2)} \sin \left(\omega_n \sqrt{1 - \xi^2} (t - t_2) \right) \quad t \geq t_2$$

Can we select the amplitudes and times of the impulses in such a way that the output is zero for $t > t_2$?

The two sinusoidal responses can be added using the following relation

$$A_1 \sin \left(\alpha t + \phi_1 \right) + A_2 \sin \left(\alpha t + \phi_2 \right) = A_t \sin \left(\alpha t + \psi \right)$$

where

$$A_t = \sqrt{\left(A_1 \cos \phi_1 + A_2 \cos \phi_2 \right)^2 + \left(A_1 \sin \phi_1 + A_2 \sin \phi_2 \right)^2}$$

$$\psi = \tan^{-1} \left(\frac{A_1 \cos \phi_1 + A_2 \cos \phi_2}{A_1 \sin \phi_1 + A_2 \sin \phi_2} \right)$$

In order to have zero output we should thus impose

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 = 0 \qquad A_1 \sin \phi_1 + A_2 \sin \phi_2 = 0$$

Applying the previous relations we obtain the following conditions

$$k_1 e^{\xi \omega_n t_1} \sin\left(\omega_n \sqrt{1-\xi^2}\, t_1\right) + k_2 e^{\xi \omega_n t_2} \sin\left(\omega_n \sqrt{1-\xi^2}\, t_2\right) = 0$$

$$k_1 e^{\xi \omega_n t_1} \cos\left(\omega_n \sqrt{1-\xi^2}\, t_1\right) + k_2 e^{\xi \omega_n t_2} \cos\left(\omega_n \sqrt{1-\xi^2}\, t_2\right) = 0$$

that can be solved with respect to k_1, k_2, t_1 and t_2.
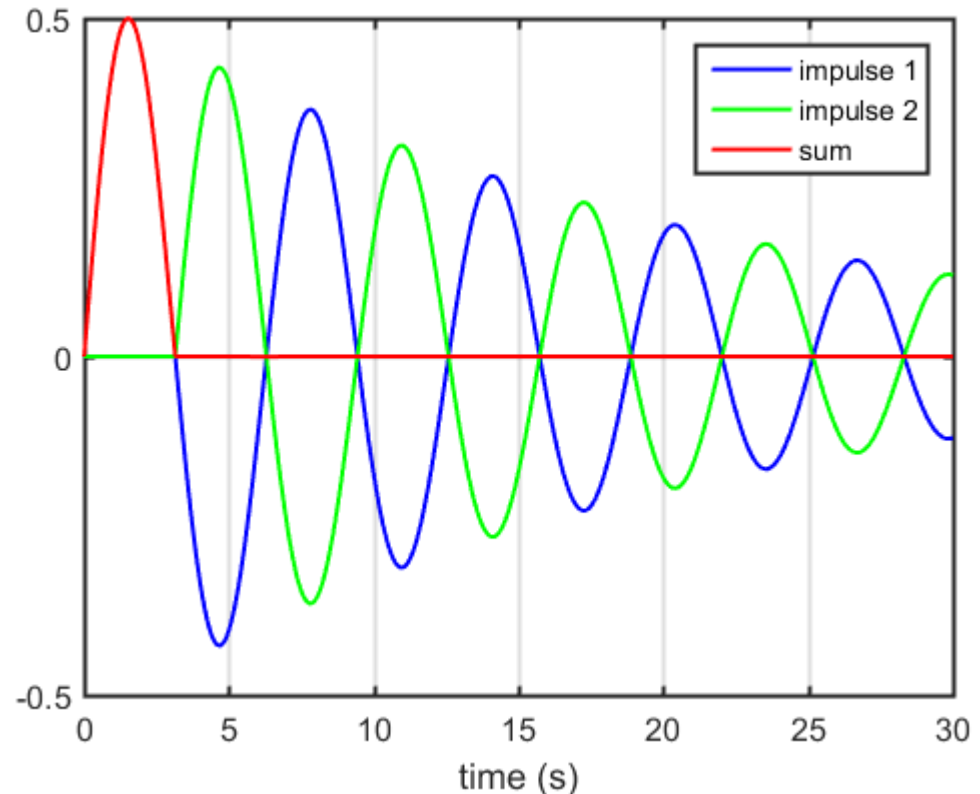
As we have two equations and four unknowns, we can set $t_1 = 0$ and introduce a normalization condition on the amplitudes

$$k_1 + k_2 = 1$$

Solving the two conditions we get

$$t_1 = 0, \quad t_2 = \frac{\Delta T}{2}, \quad k_1 = \frac{1}{1+\alpha}, \quad k_2 = \frac{\alpha}{1+\alpha} \quad \left(\alpha = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \right)$$
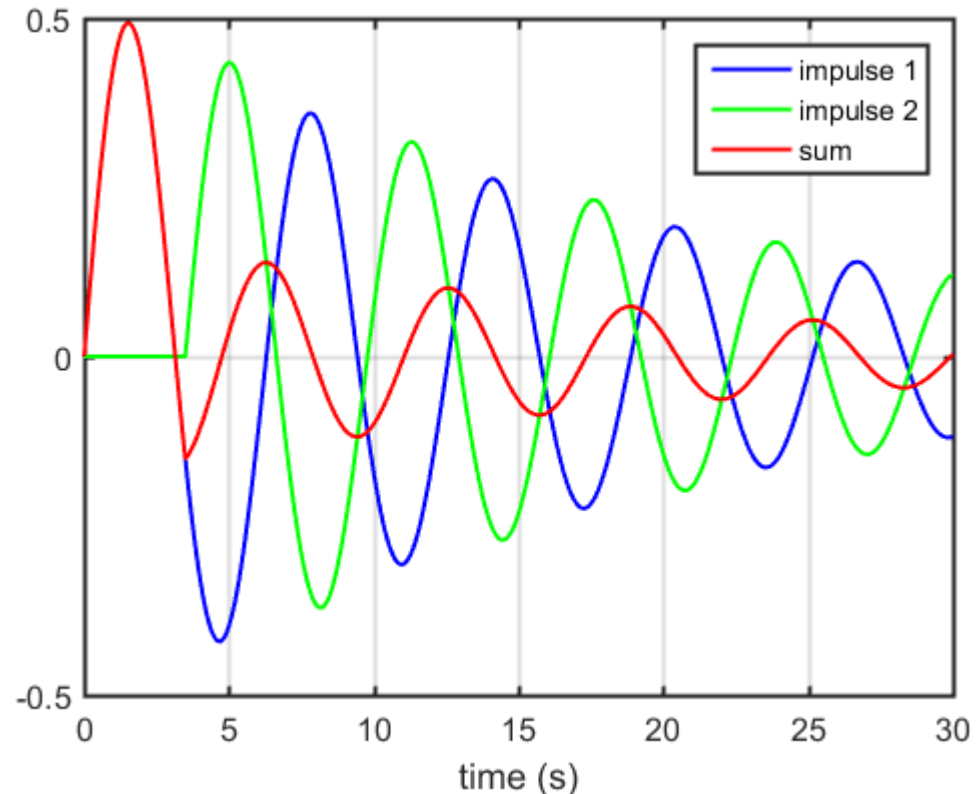
Note that both impulses are positive, and the second one is delayed of half a period.

What happens if ω_n and ξ are not perfectly known?
Assume that there is an error of 10% on ω_n and of 20% on ξ.

In order to improve the robustness with respect to errors in ω_n and ξ we have to introduce more impulses.

Let's consider now the response of the system to <u>three impulses</u>, at t_1, t_2 and t_3 ($t_3 > t_2 > t_1$). The response of the system is the sum of the three impulse responses

$$y(t) = \sum_{i=1}^{3} B_i e^{-\xi\omega_n(t-t_i)} \sin\left(\omega_n\sqrt{1-\xi^2}\,(t-t_i)\right) \quad t \geq t_3$$

The trigonometric relation can be extended to more than two impulses as follows

$$A_t = \sqrt{\left(\sum_{j=1}^{n} A_j \cos\phi_j\right)^2 + \left(\sum_{j=1}^{n} A_j \sin\phi_j\right)^2}$$

As a consequence, in order to have the output equal to zero for $t > t_3$ we impose the following conditions

$$\sum_{i=1}^{3} k_i e^{\xi\omega_n t_i} \sin\left(\omega_n\sqrt{1-\xi^2}\,t_i\right) = 0$$

$$\sum_{i=1}^{3} k_i e^{\xi\omega_n t_i} \cos\left(\omega_n\sqrt{1-\xi^2}\,t_i\right) = 0$$

As we have two equations and six unknowns, we can add a new constraint imposing that even the output derivative is zero for $t > t_3$.

In this way we obtain

$$\sum_{i=1}^{3} k_i t_i e^{\xi \omega_n t_i} \sin \left(\omega_n \sqrt{1 - \xi^2} t_i \right) = 0$$

$$\sum_{i=1}^{3} k_i t_i e^{\xi \omega_n t_i} \cos \left(\omega_n \sqrt{1 - \xi^2} t_i \right) = 0$$

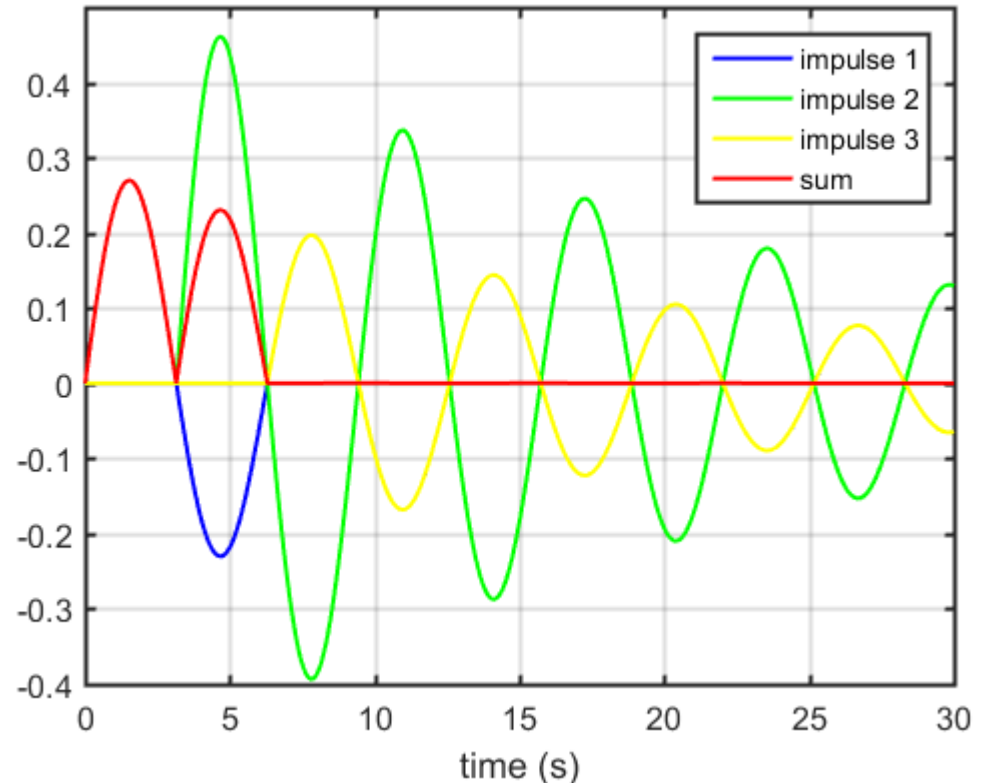and we have four equations and six unknowns.

We can finally set $t_1 = 0$ and introduce a normalization condition on the amplitudes

$$k_1 + k_2 + k_3 = 1$$

Solving the four conditions we get

$$t_1 = 0, \quad t_2 = \frac{\Delta T}{2}, \quad t_3 = \Delta T, \quad k_1 = \frac{1}{1+2\alpha+\alpha^2}, \quad k_2 = \frac{2\alpha}{1+2\alpha+\alpha^2}, \quad k_3 = \frac{\alpha^2}{1+2\alpha+\alpha^2}$$

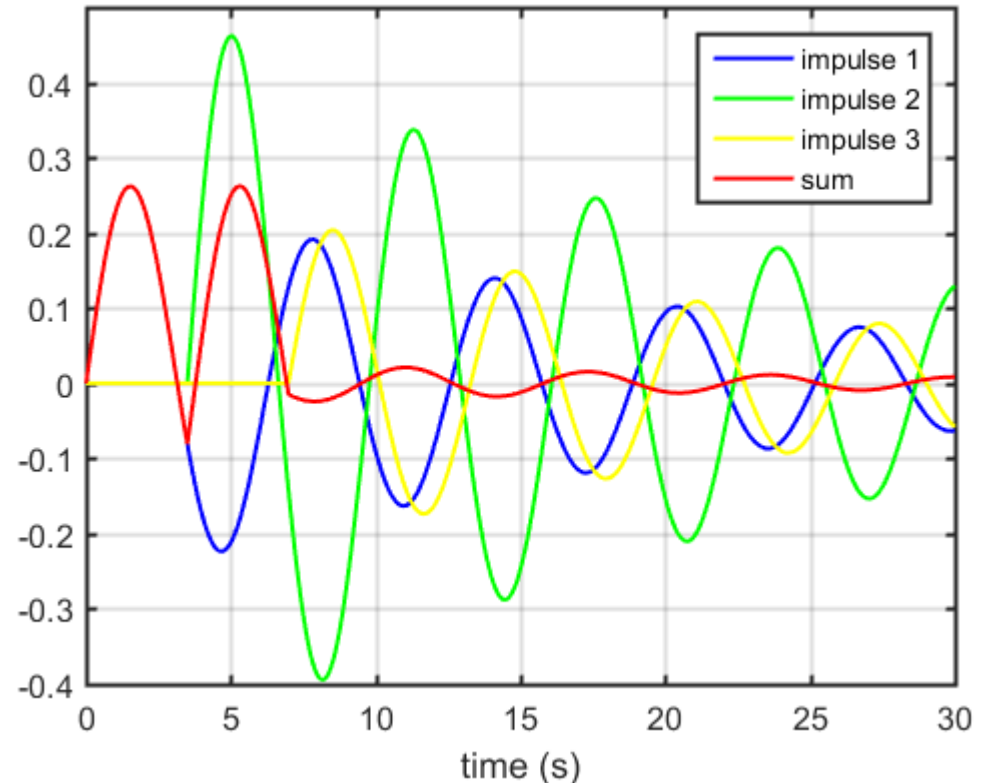Note that all impulses are positive. The second impulse is delayed of half a period, the third one of a period.

What happens if ω_n and ξ are not perfectly known?

Assume that there is an error of 10% on ω_n and of 20% on ξ.

Now we have a good damping of oscillations even in the presence of parameter errors.

We would now generalize the previous results in order to devise a method to modify the input of a resonance system in such a way that output oscillations are cancelled out.

Assuming that $u(t)$ is the system input and $w(t)$ the impulse train

$$w(t) = k_1 \delta(t) + k_2 \delta\left(t - \frac{\Delta T}{2}\right) + k_3 \delta(t - \Delta T)$$

$h(t)$ is the response of the system to the impulse train

$$h(t) = y_1(t) + y_2(t) + y_3(t), \qquad h(t) = 0 \quad t > \Delta T$$

Remember that we define convolution between two signals $u(t)$ and $h(t)$ the following operation

$$h(t) * u(t) = \int_0^t h(\tau)u(t-\tau)\mathrm{d}\tau = \int_0^t u(\tau)h(t-\tau)\mathrm{d}\tau$$

whose Laplace transform is given by

$$\mathscr{L}\{h(t) * u(t)\} = H(s)U(s)$$

Convolving the original input $u(t)$ with signal $h(t)$ we obtain a signal $y(t)$ that has no oscillations for $t > \Delta T$.
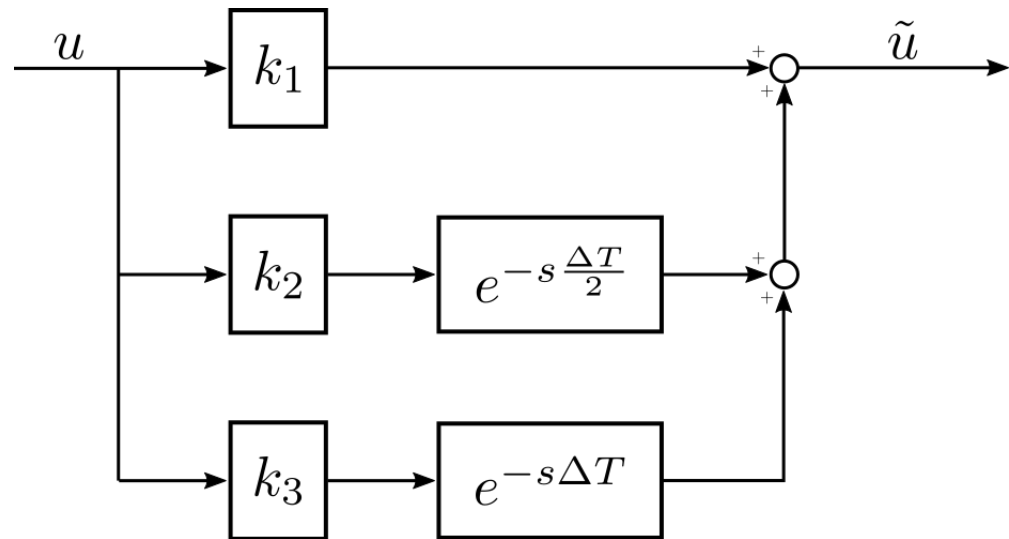
In fact, we have

$$Y(s) = \mathscr{L}\{h(t) * u(t)\} = H(s)U(s) = G(s)W(s)U(s)$$

$$= G(s)\left(k_1 + k_2 e^{-s\frac{\Delta T}{2}} + k_3 e^{-s\Delta T}\right)U(s)$$

This relation is equivalent to filter the input signal $u(t)$ with a system described by the following block diagram.
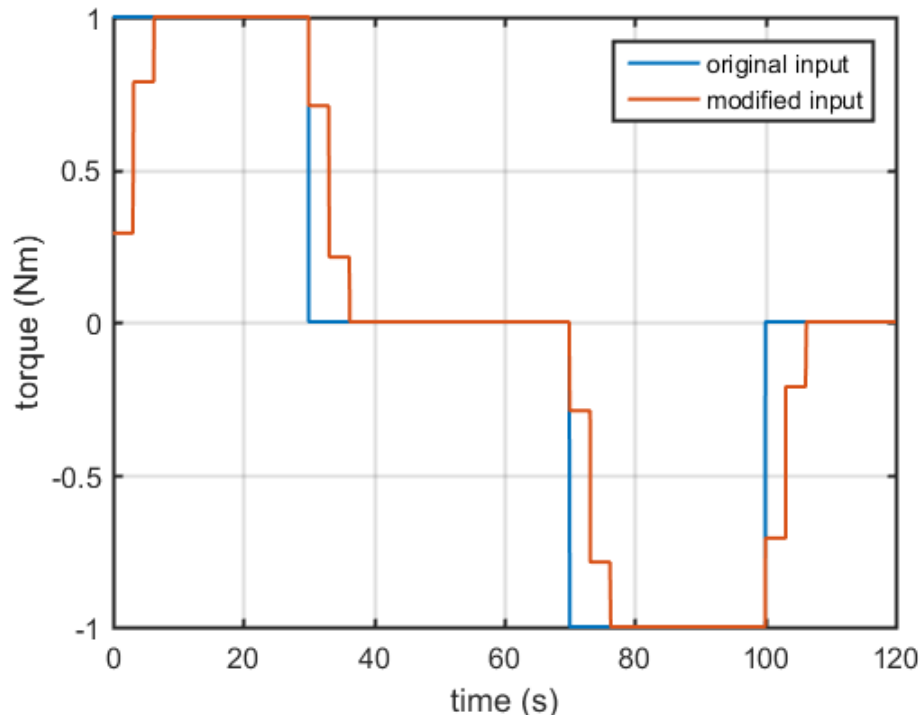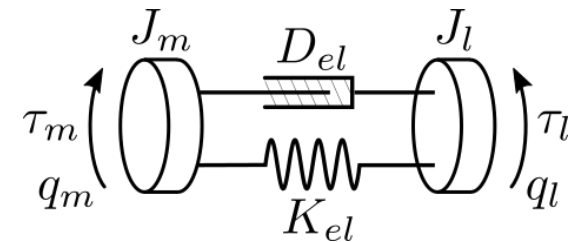
This method can be extended to systems with more than one resonance mode.
Adaptive versions exist as well.

Let's apply input shaping to an elastic servomechanism characterized by $\omega_n = 1$ and $\xi = 0.05$.

Assuming that the system input is a rectangular torque profile, we can compute coefficients k_i and time delays.

We obtain the following modified input

Let's simulate the response of the system to the rectangular torque profile, focusing on the load side velocity.

Assume that there is an error of 10% on ω_n and of 20% on ξ.

Though the uncertainty on nominal frequency and damping, the oscillations have been almost completely cancelled out.

There is a delay in the velocity that can be compensated shifting the reference signal.