Control of Mobile Robots
Exercise 2: Kinematics and Dynamics
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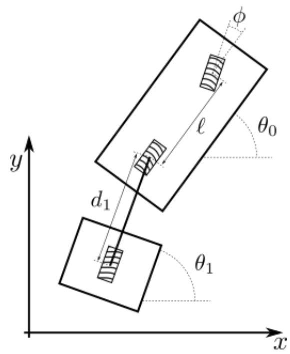
Exercise 1 - Car pulling many trailers

Consider again the car-trailer system, assuming now that the car is followed by N trailers.
Write the kinematic model of the system.
The car position (x, y) is represented by the rear wheel contact point, the position of the i-th trailer (x_{t_i}, y_{t_i}), instead, is represented by the trailer wheel contact point.

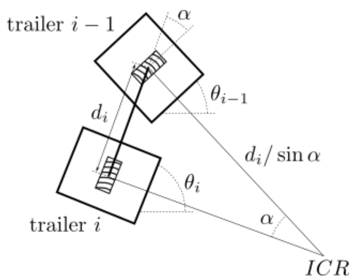


Solution

Recall the kinematic model of the car pulling a single trailer (see Exercise 1 on Kinematics) given as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta}_0 \\ \dot{\theta}_1 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \\ \dfrac{1}{\ell}\tan\phi \\ \dfrac{1}{d_1}\sin(\theta_0 - \theta_1) \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

and recall that θ_0 is the orientation of the car and v is the linear velocity of its rear wheel.
Now let us consider two arbitrary consecutive trailers, i and $i-1$. Making reference to the figure, it is possible to deduce that



$$\begin{aligned} \alpha &= \theta_{i-1} - \theta_i \\ v_i &= v_{i-1}\cos(\alpha) \\ &= v_{i-1}\cos(\theta_{i-1} - \theta_i) \end{aligned} \tag{1}$$

for all $i \geq 1$ and where $v_0 = v$.
Furthermore, from the definition of ICR, the vehicle is rotating about this point with velocity $\dot{\theta}_i$. Then we can obtain the following relation between the linear velocity, v_{i-1}, of the point of contact of wheel $i-1$ and the angular velocity $\dot{\theta}_i$

$$\dot{\theta}_i \frac{d_i}{sin(\theta_{i-1} - \theta_i)} = v_{i-1}$$

Then, the angular velocity is

$$\dot{\theta}_i = v_{i-1}\frac{\sin(\theta_{i-1} - \theta_i)}{d_i} \tag{2}$$

Taking (1) into consideration, (2) can be rewritten as

$$\dot{\theta}_i = \frac{1}{d_i} v \prod_{j=1}^{i-1} (\cos(\theta_{j-1} - \theta_j)) \sin(\theta_{i-1} - \theta_i)$$

Finally, the kinematic model of a car pulling N trailers can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta}_0 \\ \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_i \\ \vdots \\ \dot{\theta}_N \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \\ \frac{1}{\ell}\tan\phi \\ \frac{1}{d}\sin\left(\theta_0 - \theta_1\right) \\ \vdots \\ \frac{1}{d_i}\prod_{j=1}^{i-1}\left(\cos(\theta_{j-1} - \theta_j)\right)\sin(\theta_{i-1} - \theta_i) \\ \vdots \\ \frac{1}{d_N}\prod_{j=1}^{N-1}\left(\cos(\theta_{j-1} - \theta_j)\right)\sin(\theta_{N-1} - \theta_N) \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \omega$$

Note that, we could also achieve the same result by writing the pure rolling constraints. For a generic i-th trailer, the pure rolling constraint is written as

$$\dot{x}_i \sin(\theta_i) - \dot{y}_i \cos(\theta_i) = 0,\ \forall i = 0, 1, \ldots, N \tag{3}$$

where

$$x_i = x - \sum_{j=1}^{i} d_j \cos(\theta_j) \qquad y_i = y - \sum_{j=1}^{i} d_j \sin(\theta_j)$$

The time derivatives of x_i and y_i are then calculated as

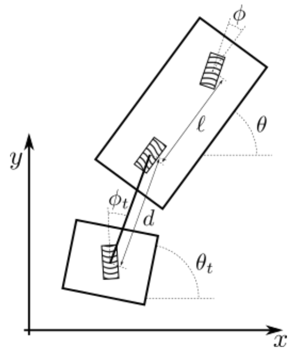
$$\dot{x}_i = \dot{x} + \sum_{j=1}^{i} d_j \sin(\theta_j)\dot{\theta}_j \qquad \dot{y}_i = \dot{y} - \sum_{j=1}^{i} d_j \cos(\theta_j)\dot{\theta}_j \tag{4}$$

Plugging (4) into (3) we can obtain the following constraint for i-th trailer

$$\dot{x}\sin(\theta_i) - \dot{y}\cos(\theta_i) + \sum_{j=1}^{i} d_j \cos(\theta_i - \theta_j)\dot{\theta}_j = 0,\ \forall i = 0, 1, \ldots, N$$

Exercise 2 - Car-trailer system, with steering trailer

Write the kinematic model of a car-trailer system. The trailer is provided with steerable wheel.
The car position (x, y) is represented by the rear wheel contact point, the trailer position (x_t, y_t), instead, is represented by the trailer wheel contact point.



Solution

The car-trailer robot configuration is represented by vector

$$\mathbf{q} = [x,\ y,\ \theta,\ \theta_t,\ \phi,\ \phi_t]^T$$

and we can write the pure rolling constraints referred to each wheel of the car and of the trailer as follows

$$\dot{x}_1 \sin(\theta + \phi) - \dot{y}_1 \cos(\theta + \phi) = 0 \tag{5}$$
$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0 \tag{6}$$
$$\dot{x}_t \sin(\theta_t + \phi_t) - \dot{y}_t \cos(\theta_t + \phi_t) = 0 \tag{7}$$

where (x_1, y_1) is the position of the car front wheel contact point.
We can relate the position of the front wheel contact point and of the trailer wheel contact point to (x, y) through a rigidity constraint

$$x_1 = x + \ell \cos\theta$$
$$y_1 = y + \ell \sin\theta$$

and

$$x_t = x - d \cos\theta_t$$
$$y_t = y - d \sin\theta_t$$

Differentiating the two relations with respect to time we obtain

$$\dot{x}_1 = \dot{x} - \ell\dot{\theta} \sin\theta$$
$$\dot{y}_1 = \dot{y} + \ell\dot{\theta} \cos\theta$$

and

$$\dot{x}_t = \dot{x} + d\dot{\theta}_t \sin\theta_t$$
$$\dot{y}_t = \dot{y} - d\dot{\theta}_t \cos\theta_t$$

Substituting these relations in (5) and (7) we obtain

$$\dot{x} \sin(\theta + \phi) - \dot{y} \cos(\theta + \phi) - \ell\dot{\theta} \cos\phi = 0$$
$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0$$
$$\dot{x} \sin(\theta_t + \phi_t) - \dot{y} \cos(\theta_t + \phi_t) - d\dot{\theta}_t \cos\phi_t = 0$$

The three constraints that describe the car-trailer can be written in Pfaffian form as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos\phi & 0 & 0 \\ \sin\theta & -\cos\theta & 0 & 0 & 0 \\ \sin(\theta_t+\phi_t) & -\cos(\theta_t+\phi_t) & 0 & -d\cos\phi_t & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\theta}_t \\ \dot{\phi} \\ \dot{\phi}_t \end{bmatrix} = 0$$

A basis for the null space of $A^T(\mathbf{q})$ is given by

$$\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{\ell}\tan\phi \\ \frac{1}{d}\frac{\sin(\theta_t + \phi_t - \theta)}{\cos\phi_t} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Finally, the kinematic model for the car-trailer is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\theta}_t \\ \dot{\phi} \\ \dot{\phi}_t \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{\ell}\tan\phi \\ \frac{1}{d}\frac{\sin(\theta_t + \phi_t - \theta)}{\cos\phi_t} \\ 0 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega_t$$

where v is the velocity of the rear wheel of the car, and ω, ω_t are the steering velocities for the front wheel of the car and the trailer wheel, respectively.

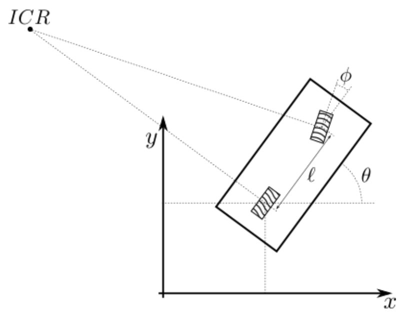
Exercise 3 - Problem 11.7 from Robotics textbook

For a generic configuration of a bicycle mobile robot, determine the Cartesian position of the instantaneous center of rotation, and derive the expression of the angular velocity of the body as a function of the robot configuration $\mathbf{q}$ and of the modulus of the velocity of rear wheel center. In the particular case of the rear-wheel drive bicycle, show that such expression is consistent with the evolution of θ predicted by the kinematic model. Moreover, compute the velocity v_P of a generic point P on the robot chassis.

Solution

The bicycle robot configuration is represented by vector

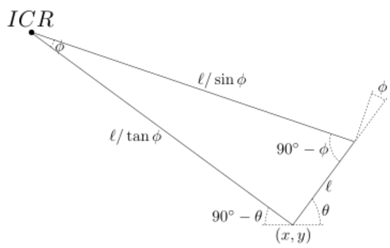
$$\mathbf{q} = [x,\ y,\ \theta,\ \phi]^T$$



The ICR is the intersection between the null velocity lines of the front and rear wheels, as shown in the picture. Moreover, straightforward geometry considerations allow to compute the position of ICR (make reference to the figure) with respect to the x, y frame, as follows

$$x_{ICR} = x - \frac{\ell}{\tan\phi}\sin\theta \qquad y_{ICR} = y + \frac{\ell}{\tan\phi}\cos\theta$$

According to the definition of ICR the vehicle is rotating around this point at velocity $\dot{\theta}$. As a consequence, if v is the linear velocity of the rear wheel contact point, the following relation holds



$$\dot{\theta}\frac{\ell}{\tan\phi} = v$$

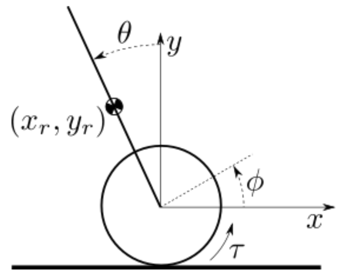
The angular velocity of the body is thus

$$\dot{\theta} = v\frac{\tan\phi}{\ell}$$

and it is straightforward to verify that, in the particular case of the rear-wheel drive bicycle, such expression is consistent with the evolution of θ predicted by the kinematic model.
A generic point P on the robot chassis rotates at a velocity $\dot{\theta}$ around the ICR. Its linear velocity is thus the product of $\dot{\theta}$ by the distance of P from the ICR.

Exercise 4 - Dynamic model of a Segway

Write the dynamic model of the Segway shown in the picture using the Lagrangian formalism. Note that, the reference frame shown in the picture is a fixed reference frame.
The Segway is described by the following parameters:



- m_w, I_w, are the mass and moment of inertia of the wheel, respectively;
- m_r, I_r, are the mass and moment of inertia of the rod, respectively;
- (x_r, y_r), are the coordinates of the rod center of gravity;
- R, is the radius of the wheel;
- ℓ, is the distance of the rod center of gravity from the center of the wheel.

Solution

First of all, the Segway motion is described by the position of the wheel center and by the angular position of the wheel and the rod. For this reason the configuration vector is $\mathbf{q} = [x,\ \phi,\ \theta]$.
The kinetic energy of the wheel and the rod can be written as

$$T_w = \frac{1}{2}m_w\dot{x}^2 + \frac{1}{2}I_w\dot{\phi}^2 \tag{8}$$
$$T_r = \frac{1}{2}m_r\dot{x}_r^2 + \frac{1}{2}m_r\dot{y}_r^2 + \frac{1}{2}I_r\dot{\theta}^2 \tag{9}$$

The position of the rod center of gravity can be expressed as a function of the wheel center as

$$x_r = x - \ell\sin\theta$$
$$y_r = \ell\cos\theta$$

and differentiation with respect to time

$$\dot{x}_r = \dot{x} - \ell\dot{\theta}\cos\theta$$
$$\dot{y}_r = -\ell\dot{\theta}\sin\theta$$

Substituting the previous relations in (9), we obtain

$$T_r = \frac{1}{2}m_r\left(\dot{x} - \ell\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}m_r\left(\ell\dot{\theta}\sin\theta\right)^2 + \frac{1}{2}I_r\dot{\theta}^2$$

The potential energy of the rod can be written as

$$V_r = m_r g\ell\cos\theta$$

and the Lagrangian of the system is given by

$$L = T_w + T_r - V_r$$

Following the Lagrangian formalism the equations of motion of the Segway are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\mathbf{q}}}\right)^T - \left(\frac{\partial L}{\partial\mathbf{q}}\right)^T = S(\mathbf{q})\tau + A(\mathbf{q})\lambda$$

where τ is the torque applied to the wheel, λ a vector of Lagrange multipliers, $S(\mathbf{q})$ a matrix that maps the input into generalized forces performing work on $\mathbf{q}$, and $A(\mathbf{q})$ is the matrix of kinematic constraints.
Applying now the partial and total derivatives to the Lagrangian function we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) = \frac{d}{dt}\left((m_w + m_r)\,\dot{x} - \ell\dot{\theta}\cos\theta\right) = (m_w + m_r)\,\ddot{x} - m_r\ell\ddot{\theta}\cos\theta + m_r\ell\dot{\theta}^2\sin\theta$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{d}{dt}\left(I_w\dot{\phi}\right) = I_w\ddot{\phi}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(-m_r\ell\dot{x}\cos\theta + m_r\ell^2\dot{\theta} + I_r\dot{\theta}\right) = m_r\ell\left(\dot{x}\dot{\theta}\sin\theta - \ddot{x}\cos\theta\right) + \left(I_r + m_r\ell^2\right)\ddot{\theta}$$

and

$$\left(\frac{\partial L}{\partial x}\right) = 0$$
$$\left(\frac{\partial L}{\partial \phi}\right) = 0$$
$$\left(\frac{\partial L}{\partial \theta}\right) = m_r \ell \sin\theta \left(\dot{x}\dot{\theta} + g\right)$$

Collecting now the addenda related to $\dot{\mathbf{q}}$, $\mathbf{q}$ we can rewrite the motion model as

$$\underbrace{\begin{bmatrix} m_w + m_r & 0 & -m_r\ell\cos\theta \\ 0 & I_w & 0 \\ -m_r\ell\cos\theta & 0 & I_r + m_r\ell^2 \end{bmatrix}}_{B(\mathbf{q})} \ddot{\mathbf{q}} + \underbrace{\begin{bmatrix} m_r\ell\dot{\theta}^2\sin\theta \\ 0 \\ -m_r g\ell\sin\theta \end{bmatrix}}_{n(\mathbf{q},\dot{\mathbf{q}})} = S(\mathbf{q})\tau + A(\mathbf{q})\lambda \tag{10}$$

We now have to analyse the kinematic model of the Segway in order to derive matrix $A(\mathbf{q})$ and $S(\mathbf{q})$. The wheel is subject to a no slip constraint $\dot{x} + R\dot{\phi} = 0$, that in Pfaffian form can be rewritten as

$$A(\mathbf{q})^T\dot{\mathbf{q}} = \begin{bmatrix} 1 & R & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = 0$$

Computing a basis of the null space of $A(\mathbf{q})^T$, we can write the kinematic model as

$$\begin{bmatrix} \dot{x} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -R \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

where $u_1 = \dot{\phi}$ and $u_2 = \dot{\theta}$, and

$$G(\mathbf{q}) = \begin{bmatrix} -R & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying equation (10) by $G(\mathbf{q})$ we obtain

$$M(\mathbf{q})\,\dot{v} + m(\mathbf{q}, v) = G^T(\mathbf{q})\,S(\mathbf{q})\,\tau$$

where

$$\begin{aligned} M(\mathbf{q}) &= G^T(\mathbf{q})\,B(\mathbf{q})\,G(\mathbf{q}) \\ &= \begin{bmatrix} (m_w + m_r)\,R^2 + I_w & m_r R\ell\cos\theta \\ m_r R\ell\cos\theta & m_r\ell^2 + I_r \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} m(\mathbf{q}, v) &= G^T(\mathbf{q})\,B(\mathbf{q})\,\dot{G}(\mathbf{q})\,v + G^T(\mathbf{q})\,n(\mathbf{q}, G(\mathbf{q})\,v) \\ &= \begin{bmatrix} -m_r R\ell\dot{\theta}^2\sin\theta \\ -m_r g\ell\sin\theta \end{bmatrix} \end{aligned}$$

where $v = \left[\dot{\phi},\ \dot{\theta}\right]^T$.

Considering that τ is the torque applied to the wheel, matrix $S(\mathbf{q})$ can be defined as $S(\mathbf{q}) = [0,\ 1,\ -1]^T$. As a consequence

$$G^T(\mathbf{q})\,S(\mathbf{q}) = \begin{bmatrix} -R & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Summarising, the dynamic and kinematic model of the Segway are given by

$$\begin{bmatrix} (m_w + m_r)\,R^2 + I_w & m_r R\ell\cos\theta \\ m_r R\ell\cos\theta & m_r\ell^2 + I_r \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} -m_r R\ell\dot{\theta}^2\sin\theta \\ -m_r g\ell\sin\theta \end{bmatrix} = \begin{bmatrix} \tau \\ -\tau \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{x} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -R & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix}$$

Exercise 5 - Tire-ground interaction model

The longitudinal force in the Fiala model is given by the following expression

$$F_x = \begin{cases} C_x \sigma_x \left(-1 + \dfrac{|\sigma_x|}{\sigma_{x_{sl}}} - \dfrac{\sigma_x^2}{3\sigma_{x_{sl}}^2} \right) & |\sigma_x| < \sigma_{x_{sl}} \\ -\mu F_z \operatorname{sign}(\sigma_x) & |\sigma_x| \geq \sigma_{x_{sl}} \end{cases}$$

Answer to the following questions:

1. Consider a force-slip relation for the longitudinal force that is linear up to the maximum available force (given by friction), and then saturates to the friction force. Write the expression of the longitudinal force.
2. During a curve a wheel is subject to a maximum longitudinal force $\bar{F}_x = 0.4\,\bar{F}_z$, and a maximum lateral force $\bar{F}_y = 0.3\,\bar{F}_z$, where $\bar{F}_z$ is the vertical load on the wheel.
   Find the value of the friction coefficient μ.
   Keeping constant the longitudinal force and doubling the vertical load, what is the maximum value of the lateral force?

Solution

1. The longitudinal force given by the discontinuous model can be represented using the following expression

   $$F_x = \begin{cases} -C_x \sigma_x & |\sigma_x| < \dfrac{\mu F_z}{C_x} \\ -\mu F_z \operatorname{sign}(\sigma_x) & |\sigma_x| \geq \dfrac{\mu F_z}{C_x} \end{cases}$$

   where C_x and σ_x are the cornering stiffness and slip angle, respectively, μ is the friction coefficient, and F_z the normal load on the wheel.

2. Using the friction circle constraint we get

   $$\sqrt{0.16\bar{F}_z^2 + 0.9\bar{F}_z^2} = \mu \bar{F}_z$$

   and solving for μ

   $$\mu = 0.5$$

   Doubling the vertical load we get

   $$\sqrt{0.16\bar{F}_z^2 + \alpha^2 \bar{F}_z^2} \leq 2 \cdot 0.5\bar{F}_z = \bar{F}_z$$

   and solving for α

   $$\alpha^2 \leq 1 - 0.16 = 0.84$$

   The maximum lateral force becomes approximately $0.92\,\bar{F}_z$.