Automatic Control
Exercise 1: Time domain analysis of dynamical systems
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Exercise 1

Consider the following continuous time nonlinear and time invariant dynamical system

$$\begin{aligned}\dot{x}_1(t) &= x_2^2(t) + \alpha u(t)\\ \dot{x}_2(t) &= x_1(t)x_2(t) + u(t)\\ y(t) &= \beta x_1(t)\end{aligned}$$

and assume that $u(t) = \bar{u} = 1$.
Find α and β in such a way that at the equilibrium

$$\begin{aligned}x_2(t) &= \bar{x}_2 = 2\\ y(t) &= \bar{y} = 8\end{aligned}$$

and compute the value of $\bar{x}_1$.
Find another state equilibrium related to the same values of α and β.

Solution

The equilibrium equations are

$$\begin{aligned}\bar{x}_2^2 + \alpha\bar{u} &= 0\\ \bar{x}_1\bar{x}_2 + \bar{u} &= 0\\ \bar{y} &= \beta\bar{x}_1\end{aligned}$$

Assuming $\bar{x}_2 = 2$, $\bar{y} = 8$ and $\bar{u} = 1$ we obtain

$$\begin{aligned}\alpha &= -\bar{x}_2^2 = -4\\ 2\bar{x}_1 &= -1 \Rightarrow \bar{x}_1 = -\frac{1}{2}\\ \beta &= \frac{\bar{y}}{\bar{x}_1} = 8 \cdot (-2) = -16\end{aligned}$$

We can now determine the other equilibrium corresponding to $\alpha = -4$ and $\beta = -16$

$$\begin{aligned}\bar{x}_2^2 &= 4\\ \bar{x}_1\bar{x}_2 &= -1\\ \bar{y} &= -16\bar{x}_1\end{aligned}$$

from this equations it follows

$$\begin{aligned}\bar{x}_2 &= -2\\ \bar{x}_1 &= \frac{1}{2}\\ \bar{y} &= -8\end{aligned}$$

Exercise 2

Consider the following continuous time linear and time invariant dynamical system

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) + \alpha x_2(t) + u(t)\\ \dot{x}_2(t) &= \alpha x_1(t) - 2x_2(t)\\ y(t) &= x_1(t)\end{aligned}$$

Find α for which the system is asymptotically stable.
For $\alpha = 0$ determine the value of the initial condition $x(0)$ in such a way that the output response to $u(t) = e^{at}$ is $y(t) = ku(t)$ where k is a constant that has to be determined.

Solution

The system state matrix is

$$\begin{bmatrix} -2 & \alpha \\ \alpha & -2 \end{bmatrix}$$

whose characteristic polynomial is

$$\varphi(\lambda) = \lambda^2 + 4\lambda + (4 - \alpha^2)$$

Thanks to the necessary condition, that for second order systems is sufficient as well, the system is asymptotically stable if

$$4 - \alpha^2 > 0 \quad \Rightarrow \quad -2 < \alpha < 2$$

For $\alpha = 0$ the system equations become

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + u(t) \\ \dot{x}_2(t) &= -2x_2(t) \\ y(t) &= x_1(t) \end{aligned}$$

It is straightforward to notice that the system response does not depend on the second state variable. In order to compute the output we can consider the reduced system

$$\begin{aligned} \dot{x}_1(t) &= -2x_1(t) + u(t) \\ y(t) &= x_1(t) \end{aligned}$$

The output response is given by

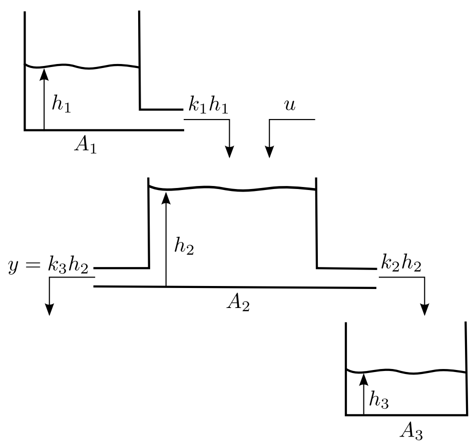
$$\begin{aligned} y(t) &= x_1(0)e^{-2t} + \int_0^t e^{-2(t-\tau)}e^{a\tau}\mathrm{d}\tau = x_1(0)e^{-2t} + e^{-2t}\int_0^t e^{(a+2)\tau}\mathrm{d}\tau = x_1(0)e^{-2t} + e^{-2t}\left[\frac{e^{(a+2)\tau}}{a+2}\right]_0^t \\ &= e^{-2t}\left[x_1(0) + \frac{e^{(a+2)t}}{a+2} - \frac{1}{a+2}\right] \end{aligned}$$

Assuming $x_1(0) = \frac{1}{a+2}$ and $k = \frac{1}{a+2}$ we obtain

$$y(t) = \frac{1}{a+2}e^{at} = ku(t)$$

Exercise 3

Given the following system of tanks



where $A_1 = A_3 = 0.5$, $A_2 = 1$, and $k_1 = k_2 = k_3 = 1$.
Find the equations of the dynamical system that describes the system of tanks.
Determine the state and output equilibria corresponding to $u(t) = \bar{u} = 0$.
Is the system stable, unstable or asymptotically stable?

Solution

The dynamical system that describes the system of tanks is given by

$$\begin{aligned}
A_1\dot{h}_1(t) &= -k_1h_1(t)\\
A_2\dot{h}_2(t) &= u(t) + k_1h_1(t) - k_3h_2(t) - k_2h_2(t)\\
A_3\dot{h}_3(t) &= k_2h_2(t)\\
y(t) &= k_3h_2(t)
\end{aligned}$$

and substituting the values of the parameters

$$\begin{aligned}
\dot{x}_1(t) &= -2x_1(t)\\
\dot{x}_2(t) &= x_1(t) - 2x_2(t) + u(t)\\
\dot{x}_3(t) &= 2x_2(t)\\
y(t) &= x_2(t)
\end{aligned}$$

The equilibrium equations are

$$\begin{aligned}
0 &= -2\bar{x}_1(t)\\
0 &= \bar{x}_1(t) - 2\bar{x}_2(t)\\
0 &= 2\bar{x}_2(t)\\
\bar{y}(t) &= \bar{x}_2(t)
\end{aligned}$$

The equilibrium point is thus characterized by $\bar{y} = \bar{x}_1 = \bar{x}_2 = 0$, $\forall \bar{x}_3$.
The state matrix is

$$\begin{bmatrix} -2 & 0 & 0\\ 1 & -2 & 0\\ 0 & 2 & 0 \end{bmatrix}$$

whose eigenvalues are -2, -2 and 0. The system is thus stable.

Exercise 4

Consider the following continuous time nonlinear and time invariant dynamical system

$$\begin{aligned}
\dot{x}(t) &= x^2(t) - u(t)x(t) - 2u(t)\\
y(t) &= x^3(t) + u^3(t)
\end{aligned}$$

and assume that $u(t) = \bar{u} = 1$.
Find the state and output equilibria and, for each of them, the linearized system.
Analyse the stability of the linearized systems and compute the response $\delta y(t)$ to an input $\delta u(t) = 0.1\cos(2t)$ for $t \geq 0$.

Solution

The equilibrium equations are

$$\begin{aligned}
0 &= \bar{x}^2 - \bar{u}\bar{x} - 2\bar{u}\\
\bar{y} &= \bar{x}^3 + \bar{u}^3
\end{aligned}$$

and considering $\bar{u} = 1$

$$\begin{aligned}
0 &= \bar{x}^2 - \bar{x} - 2\\
\bar{y} &= \bar{x}^3 + 1
\end{aligned}$$

from which we obtain

$$\bar{x} = 2 \quad \bar{y} = 9$$

and

$$\bar{x} = -1 \quad \bar{y} = 0$$

The linearized system computed for the general equilibrium $(\bar{x}, \bar{u})$ has the following expression

$$\begin{aligned} \delta\dot{x}(t) &= (2\bar{x} - \bar{u})\delta x(t) - (\bar{x} + 2)\delta u(t) \\ \delta y(t) &= 3\bar{x}^2 \delta x(t) + 3\bar{u}^2 \delta u(t) \end{aligned}$$

Considering now the first equilibrium we obtain

$$\begin{aligned} \delta\dot{x}(t) &= 3\delta x(t) - 4\delta u(t) \\ \delta y(t) &= 12\delta x(t) + 3\delta u(t) \end{aligned}$$

and for the second one

$$\begin{aligned} \delta\dot{x}(t) &= -3\delta x(t) - \delta u(t) \\ \delta y(t) &= 3\delta x(t) + 3\delta u(t) \end{aligned}$$

The first linearized system, and the related equilibrium point, are unstable; the second linearized system, and the related equilibrium point, are asymptotically stable.
Consider now the state trajectory of the first linearized system for $\delta u(t) = 0.1\cos(2t)$

$$\delta x(t) = e^{3t}\delta x_0 + \int_0^t e^{3(t-\tau)}(-4)0.1\cos(2\tau)\mathrm{d}\tau$$

and assuming $x(0) = \bar{x}$, i.e. $\delta x_0 = 0$, we obtain[1]

$$\delta x(t) = -0.4e^{3t}\int_0^t e^{-3\tau}\cos(2\tau)\mathrm{d}\tau = -\frac{4}{130}\left(2\sin(2\tau) - 3\cos(2\tau)\right)$$

Finally, the state trajectory of the second linearized system for $\delta u(t) = 0.1\cos(2t)$ is given by[2]

$$\delta x(t) = \int_0^t e^{-3(t-\tau)}(-1)0.1\cos(2\tau)\mathrm{d}\tau = -0.1e^{-3t}\int_0^t e^{3\tau}\cos(2\tau)\mathrm{d}\tau = -\frac{1}{130}\left(2\sin(2\tau) + 3\cos(2\tau)\right)$$

The output trajectories of the two linearized system are then given by

$$\delta y(t) = 12\delta x(t) + 3\delta u(t) = \frac{3}{65}\left(\frac{87}{2}\cos(2\tau) - 16\sin(2\tau)\right)$$

and

$$\delta y(t) = 3\delta x(t) + 3\delta u(t) = \frac{3}{13}\left(\cos(2\tau) - \frac{1}{5}\sin(2\tau)\right)$$

Exercise 5

Find the values α and β for which the system with characteristic polynomial

$$\varphi(s) = s^3 + \alpha s^2 + \beta s + 1$$

is asymptotically stable.
Plot the stability region in the (α, β) plain.

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[1] Note that, using integration by parts one obtains

$$\int e^{-3\tau}\cos(2\tau)\mathrm{d}\tau = \frac{1}{2}e^{3\tau}\sin(2\tau) - \frac{3}{2}\int e^{3\tau}\sin(2\tau)\mathrm{d}\tau = \frac{1}{2}e^{3\tau}\sin(2\tau) - \frac{3}{2}\left[-\frac{1}{2}e^{3\tau}\cos(2\tau) + \frac{3}{2}\int e^{3\tau}\cos(2\tau)\mathrm{d}\tau\right]$$

and thus

$$\int e^{3\tau}\cos(2\tau)\mathrm{d}\tau = \frac{1}{13}e^{3\tau}\left(2\sin(2\tau) + 3\cos(2\tau)\right)$$

[2] Note that, using integration by parts one obtains

$$\int e^{3\tau}\cos(2\tau)\mathrm{d}\tau = \frac{1}{2}e^{-3\tau}\sin(2\tau) + \frac{3}{2}\int e^{-3\tau}\sin(2\tau)\mathrm{d}\tau = \frac{1}{2}e^{-3\tau}\sin(2\tau) + \frac{3}{2}\left[-\frac{1}{2}e^{-3\tau}\cos(2\tau) - \frac{3}{2}\int e^{-3\tau}\cos(2\tau)\mathrm{d}\tau\right]$$

and thus

$$\int e^{-3\tau}\cos(2\tau)\mathrm{d}\tau = \frac{1}{13}e^{-3\tau}\left(2\sin(2\tau) - 3\cos(2\tau)\right)$$

Solution

We can build the Routh table

$$\begin{array}{ccc} 1 & \beta & 0 \\ \alpha & 1 & 0 \\ \dfrac{\alpha\beta - 1}{\alpha} & 0 & \\ 1 & & \end{array}$$

Imposing that all the elements in the first column are positive (like the first and the last one) we obtain

$$\begin{aligned} \alpha &> 0 \\ \alpha\beta - 1 &> 0 \end{aligned}$$

and thus

$$\begin{aligned} &\alpha > 0 \\ &\beta > \frac{1}{\alpha} \end{aligned}$$

The stability region in the (α, β) plain is shown in Fig. 1.

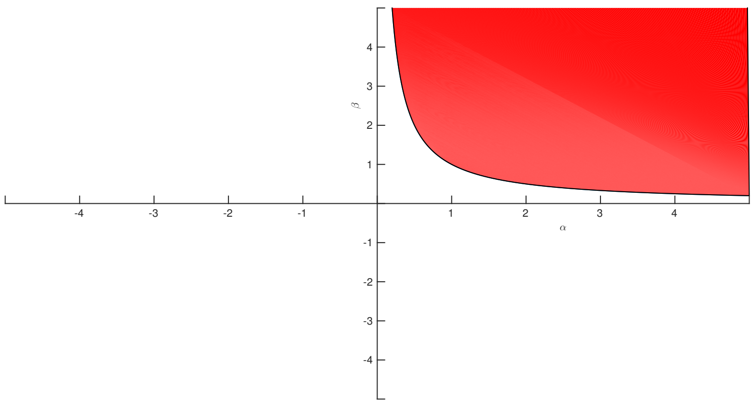

Figure 1: Stability region in the (α, β) plain.

Exercise 6

Find the values α and β for which the system with characteristic polynomial

$$\varphi(s) = s^3 + (\beta + 1)s^2 + (\beta + 1)s + (\alpha - 2\beta)$$

is asymptotically stable.
Plot the stability region in the (α, β) plain.

Solution

From the necessary condition we can derive the following constraints

$$\begin{aligned} \beta + 1 &> 0 \\ \alpha - 2\beta &> 0 \end{aligned}$$

or, equivalently

$$\begin{aligned} &\beta > -1 \\ &\beta < \frac{\alpha}{2} \end{aligned}$$

Fig. 2 shows these constraints in the (α, β) plain.

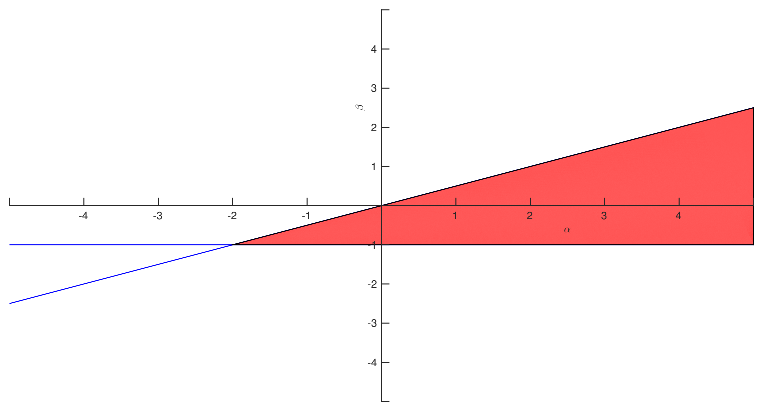

Figure 2: Stability region in the (α, β) plain.

We can now build the Routh table

$$\begin{array}{cc} 1 & \beta+1 \\ \beta+1 & \alpha-2\beta \\ \dfrac{\beta^2+4\beta+1-\alpha}{\beta+1} & \\ \alpha-2\beta & \end{array}$$

Imposing that all the elements in the first column are positive (like the first one) we obtain

$$\begin{aligned} \beta+1 &> 0 \\ \beta^2+4\beta+1-\alpha &> 0 \\ \alpha-2\beta &> 0 \end{aligned}$$

and thus

$$\begin{aligned} \beta &> -1 \\ \beta &< \frac{\alpha}{2} \\ \beta^2+4\beta+1 &> \alpha \end{aligned}$$

The stability region in the (α, β) plain is shown in Fig. 3.

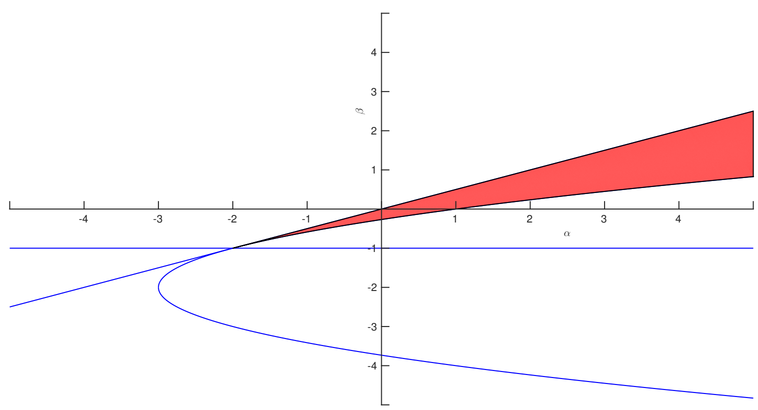

Figure 3: Stability region in the (α, β) plain.