Automatic Control
Exercise 6: Discrete time systems and digital control design
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Exercise 1

Write the dynamical system describing the population of students of an high school, assuming that:

- no students leave the school before the end of the third year;
- students attending the first year are newly enrolled students, and students that fail to be admitted to the second year;
- students attending the second year are students that were promoted at the end of the first year, and students that fail to be admitted to the third year;
- students attending the third year are students that were promoted at the end of the second year, and students that fail to be admitted to the final exam;
- we are interested to study the total number of students enrolled in the school (i.e., attending the first, second, and third year).

Compute the system transfer function.

Solution

Defining:

- $u(k)$, number of newly enrolled students at year $k+1$;
- $x_1(k)$, number of students attending the first year, at year k;
- $x_2(k)$, number of students attending the second year, at year k;
- $x_3(k)$, number of students attending the third year, at year k;
- $y(k)$, number of students enrolled in the school at year k;

we can write the following dynamical system

$$
\begin{aligned}
x_1(k+1) &= \alpha x_1(k) + u(k) \\
x_2(k+1) &= (1-\alpha)x_1(k) + \beta x_2(k) \\
x_3(k+1) &= (1-\beta)x_2(k) + \gamma x_3(k) \\
y(k) &= x_1(k) + x_2(k) + x_3(k)
\end{aligned}
$$

where α, β and γ are the fail rates at first, second and third year, respectively.
Applying now the Z-transform, assuming zero initial conditions, we obtain

$$
\begin{aligned}
(z-\alpha)X_1(z) &= U(z) \\
(z-\beta)X_2(z) &= (1-\alpha)X_1(z) \\
(z-\gamma)X_3(z) &= (1-\beta)X_2(z) \\
Y(z) &= X_1(z) + X_2(z) + X_3(z)
\end{aligned}
$$

Consequently

$$
\begin{aligned}
X_1(z) &= \frac{U(z)}{z-\alpha} \\
X_2(z) &= \frac{1-\alpha}{(z-\alpha)(z-\beta)}U(z) \\
X_3(z) &= \frac{(1-\alpha)(1-\beta)}{(z-\alpha)(z-\beta)(z-\gamma)}U(z)
\end{aligned}
$$

and, finally, we obtain

$$
G(z) = \frac{z^2 + (1-\alpha-\beta-\gamma)z + (1-\alpha-\beta+\alpha\beta+\beta\gamma)}{(z-\alpha)(z-\beta)(z-\gamma)}
$$

Exercise 2

Consider the following characteristic polynomial of a discrete time system

$$\varphi(z) = 3z^2 + z + \alpha \qquad \alpha \in \mathbb{R}$$

Find the values of α, without computing the roots of $\varphi(z)$, for which all the roots have magnitude less than 1.

Solution

Applying the bilinear transformation

$$z = \frac{1+s}{1-s}$$

one obtains

$$3\frac{(1+s)^2}{(1-s)^2} + \frac{1+s}{1-s} + \alpha = 0 \qquad \Rightarrow \qquad 3(1+s)^2 + (1+s)(1-s) + \alpha(1-s)^2 = (2+\alpha)s^2 + (6-2\alpha)s + (4+\alpha) = 0$$

Thanks to the bilinear transformation, we have now to find the values of α for which the s-polynomial has all roots in the open left half plane. We can thus apply the necessary and sufficient condition, obtaining

$$\begin{cases} 2+\alpha > 0 \\ 6-2\alpha > 0 \\ 4+\alpha > 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} \alpha > -2 \\ \alpha < 3 \\ \alpha > -4 \end{cases}$$

or

$$\begin{cases} 2+\alpha < 0 \\ 6-2\alpha < 0 \\ 4+\alpha < 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} \alpha < -2 \\ \alpha > 3 \\ \alpha < -4 \end{cases}$$

The second set of inequalities have no solution, from the first one, instead, we obtain $-2 < \alpha < 3$.

Exercise 3

Find the state and output equilibria of the discrete time system

$$\begin{aligned} x_1(k+1) &= x_1(k) + (1 - x_1(k))\,(1 + x_2(k)) + u(k) \\ x_2(k+1) &= x_1(k) + (1 + x_1(k))\,(1 - x_2(k)) - u(k) \\ y(k) &= (x_1(k) + x_2(k))^3 \end{aligned}$$

for $u(k) = \bar{u} = 0$. Compute the linearised system associated to each equilibrium and use it to assess the stability of the equilibrium point.

Solution

The state and output equilibria are given by

$$\begin{aligned} \bar{x_1} &= \bar{x_1} + (1 - \bar{x_1})(1 + \bar{x_2}) \\ \bar{x_2} &= \bar{x_1} + (1 + \bar{x_1})(1 - \bar{x_2}) \\ \bar{y} &= (\bar{x_1} + \bar{x_2})^3 \end{aligned}$$

solving this linear system we obtain $\bar{x}_1 = \bar{x}_2 = 1$ with output $\bar{y} = 8$, and $\bar{x}_1 = \bar{x}_2 = -1$ with output $\bar{y} = -8$. The linearised system, around the general equilibrium $\bar{x}_1, \bar{x}_2, \bar{u}$ is given by

$$\begin{aligned} \delta x_1(k+1) &= -\bar{x}_2 \delta x_1(k) + (1 - \bar{x}_1)\delta x_2(k) + \delta u(k) \\ \delta x_2(k+1) &= (2 - \bar{x}_2)\delta x_1(k) - (1 + \bar{x}_1)\delta x_2(k) - \delta u(k) \\ \delta y(k) &= 3(\bar{x}_1 + \bar{x}_2)^2 \delta x_1(k) + 3(\bar{x}_1 + \bar{x}_2)^2 \delta x_2(k) \end{aligned}$$

Consider now the equilibrium $\bar{x}_1 = \bar{x}_2 = 1$, the linearised system is

$$\begin{aligned} \delta x_1(k+1) &= -\delta x_1(k) + \delta u(k) \\ \delta x_2(k+1) &= \delta x_1(k) - 2\delta x_2(k) - \delta u(k) \\ \delta y(k) &= 12\delta x_1(k) + 12\delta x_2(k) \end{aligned}$$

from which we get the following state matrix

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

The eigenvalues are -1, -2, and we thus conclude that the equilibrium point is unstable.
Consider now the equilibrium $\bar{x}_1 = \bar{x}_2 = -1$, the linearised system is

$$\begin{aligned} \delta x_1(k+1) &= \delta x_1(k) + 2\delta x_2(k) + \delta u(k) \\ \delta x_2(k+1) &= 3\delta x_1(k) - \delta u(k) \\ \delta y(k) &= 12\delta x_1(k) + 12\delta x_2(k) \end{aligned}$$

from which we get the following state matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

The characteristic polynomial is

$$\varphi(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -3 & \lambda \end{vmatrix} = \lambda^2 - \lambda - 6$$

The eigenvalues are 3, -2, and we thus conclude that the equilibrium point is unstable.

Exercise 4

Find the analytical expression of the impulse response of the following discrete time dynamical system

$$G(z) = \frac{z-b}{z-a}$$

and verify the result computing the first 3 values of the response using the long division.

Solution

The Z-transform of the output is ($U(z) = 1$)

$$Y(z) = \frac{z-b}{z-a}$$

Applying Heaviside decomposition to $Y(z)/z$ we obtain

$$\frac{Y(z)}{z} = \frac{z-b}{z(z-a)} = \frac{\alpha}{z} + \frac{\beta}{z-a} = \frac{(\alpha+\beta)z - \alpha a}{z(z-a)}$$

and thus

$$\alpha + \beta = 1, \quad -\alpha a = -b \qquad \Rightarrow \qquad \alpha = \frac{b}{a}, \quad \beta = \frac{a-b}{a}$$

The Z-transform of the output can be expressed as

$$Y(z) = \frac{b}{a} + \frac{a-b}{a}\frac{z}{z-a}$$

and antitransforming

$$y(k) = \frac{b}{a}\text{imp}(k) + \frac{a-b}{a}a^k \qquad k \geq 0$$

Evaluating this relation for $k = 0, 1, 2$ we obtain

$$y(0) = 1, \quad y(1) = a - b, \quad y(2) = a(a-b)$$

Applying now the long division

$$\frac{z-b}{z-a} = 1 + (a-b)z^{-1} + a(a-b)z^{-2} + \dots$$

that is in accordance with the previous result.

Exercise 5

Given the following discrete time system

$$G(z) = \frac{20}{1+10z}$$

compute the analytical expression of the step response, the first 4 samples using the long division and the steady state value using, if it is possible, the final value theorem.

Solution

The Z-transform of the step response is given by

$$Y(z) = \frac{20z}{(1+10z)(z-1)} = \frac{2z}{(z+0.1)(z-1)}$$

The Heaviside decomposition is

$$\frac{Y(z)}{z} = \frac{2}{(z+0.1)(z-1)} = \frac{\alpha}{z+0.1} + \frac{\beta}{z-1} = \frac{(\alpha+\beta)z + (0.1\beta - \alpha)}{(z+0.1)(z-1)}$$

giving rise to the following relations between the two parameters

$$\alpha + \beta = 0, \quad 0.1\beta - \alpha = 2 \qquad \Rightarrow \qquad \beta \approx 1.8, \quad \alpha \approx -1.8$$

The Z-transform of the step response can be thus rewritten as

$$Y(z) = -1.8\frac{z}{z+0.1} + 1.8\frac{z}{z-1}$$

and antitransforming

$$y(k) = -1.8(-0.1)^k + 1.8 \qquad k \geq 0$$

The first four samples of the step response can be obtained applying the long division

$$Y(z) = \frac{20z}{z^2+0.9z-0.1} = 2z^{-1} - 1.8z^{-2} + 1.82z^{-3} + \dots$$

from which we derive

$$y(0) = 0, \quad y(1) = 2, \quad y(2) = -1.8, \quad y(3) = 1.82$$

Finally, as the system is asymptotically stable we can apply the final value theorem to compute the steady state value of the step response

$$\lim_{k\to\infty} y(k) = \lim_{z->1} (z-1)\frac{2z}{(z+0.1)(z-1)} = \lim_{z->1} \frac{2z}{z+0.1} \approx 1.8$$

Note that, this is in accordance with the analytical expression of the step response.

Exercise 6

Given the following transfer functions

$$G_1(z) = \frac{z-1}{z-0.5} \qquad G_2(z) = \frac{(z-1)(z-2)}{z-0.5}$$

find the correspondent difference equations and comment the results.

Solution

We can rewrite the two transfer functions as follows

$$Y(z) = G_1(z)U(z) = \frac{z-1}{z-0.5}U(z) = \frac{1-z^{-1}}{1-0.5z^{-1}}U(z)$$

$$Y(z) = G_2(z)U(z) = \frac{(z-1)(z-2)}{z-0.5}U(z) = \frac{1-3z^{-1}+2z^{-2}}{z^{-1}-0.5z^{-2}}U(z)$$

and applying the Z-transform properties we obtain the correspondent difference equations

$$y(k) = 0.5y(k-1) + u(k) - u(k-1)$$

$$y(k) = 0.5y(k-1) + u(k+1) - 3u(k) + 2u(k-1)$$

In the first equation, that corresponds to a causal transfer function, the output at time k is a function of the actual and past values of the input u, and the past values of the output itself. In the second equation, that corresponds to an a-causal transfer function, the output at time k is not only a function of the actual and past values of the input u, and the past values of the output itself, but it also depends on the future values of the output making this relation not suitable for a real-time implementation.

Exercise 7

Given the process

$$G(s) = \frac{1}{s(s+1.5)}$$

and the regulator

$$R(s) = 1.5$$

find a sample time ensuring a reduction of the phase margin equal to 4°.

Solution

The loop transfer function is

$$L(s) = \frac{1.5}{s(s+1.5)}$$

It is straightforward to notice that the crossover frequency is equal to $1\,rad/s$.
The reduction of the phase margin due to the introduction of the sampler and holder is

$$\omega_c \frac{T_s}{2}\frac{180°}{\pi} = 4° \qquad \Rightarrow \qquad T_s = 4° \frac{\pi}{180°}\frac{2}{\omega_c} = 0.14\,s$$

Exercise 8

Given the process

$$G(s) = \frac{1}{s+1}$$

and the regulator

$$R°(s) = \frac{0.8}{s}$$

compute the transfer function of the digital regulator using forward Euler method with $T_s = 0.1\,s$, and write the correspondent difference equation.

Solution

The transfer function of the digital regulator is given by

$$R(z) = R°\left(\frac{z-1}{T_s}\right) = \frac{0.8}{\frac{z-1}{0.1}} = \frac{0.08}{z-1}$$

The difference equation representing the digital regulator is

$$(1-z^{-1})U(z) = 0.08z^{-1}E(z) \qquad \Rightarrow \qquad u(k) = u(k-1) + 0.08e(k-1)$$

Exercise 9

Given the following continuous time signal

$$y(t) = 10 + 2\sin(5t) + \sin(25t)$$

Answer to the following questions:

- find the maximum sampling time T_s for which there is no aliasing effect;
- assuming $T_s = 0.1\pi\,s$ find the frequency of the aliasing harmonics;
- assuming $T_s = 0.1\pi\,s$ design an anti-aliasing filter for the signal

$$\tilde{y}(t) = 10 + 2\sin(5t) + \sin(200t)$$

Solution

The maximum frequency of $y(t)$ is $25\,rad/s$. In order to have $\Omega_N > 25\,rad/s$ we must satisfy the following inequality

$$\frac{\pi}{T_s} > 25 \qquad \Rightarrow \qquad T_s < \frac{\pi}{25} \approx 0.126\,s$$

Assuming now $T_s = 0.1\pi\,s$, i.e., $\Omega_N = 10\,rad/s$ and $\Omega_s = 20\,rad/s$, the aliasing effect is generated by the harmonic of $y(t)$ at $25\,rad/s$.
As can be seen in Fig. 1 the aliasing harmonic is at frequency $5\,rad/s$.

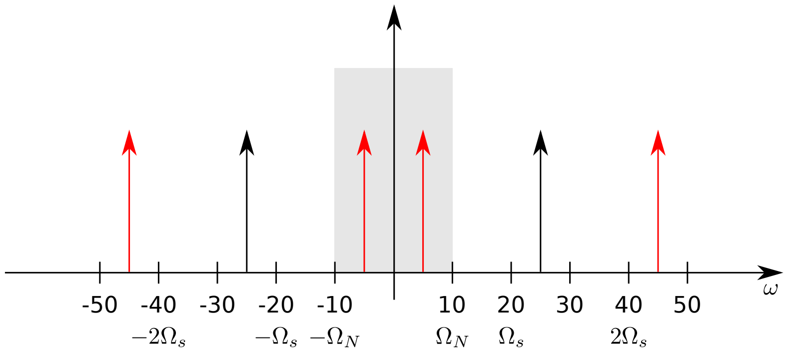

Figure 1: Aliasing harmonics.

Assuming again $T_s = 0.1\pi\,s$, i.e., $\Omega_N = 10\,rad/s$ and $\Omega_s = 20\,rad/s$, the aliasing effect is generated by the harmonic of $\tilde{y}(t)$ at $200\,rad/s$. In order to have an attenuation of $20\,dB$ at the Nyquist frequency we select a first order anti-aliasing filter with cut-off frequency at $1\,rad/s$. The filter transfer function is thus

$$F(s) = \frac{1}{1+s}$$

Exercise 10

Given the following linear and time invariant dynamical system

$$G(s) = \frac{5}{1+10s}$$

compute the correspondent discrete time transfer function using the zero-order hold method.

Solution

In order to compute the discrete time transfer function using the zero-order hold method we can adopt the following procedure:

1. we compute the analytical expression $y(t)$ of the step response;
2. we sample the continuous time step response, obtaining the digital signal $y^*(k) = y(kT_s)$;
3. we apply the Z-transform to the digital signal, obtaining $Y^*(z)$;
4. we compute the discrete time transfer function as

$$G^*(z) = Y^*(z)\frac{z-1}{z}$$

The Laplace transform of the step response is given by

$$Y(s) = \frac{G(s)}{s} = \frac{5}{s(1+10s)} = \frac{0.5}{s(s+0.1)} = \frac{5}{s} - \frac{5}{s+0.1}$$

Antitransforming we obtain

$$y(t) = 5\left(1 - e^{-0.1t}\right) \qquad t \geq 0$$

Sampling this signal with period T_s yields

$$y^*(k) = y(kT_s) = 5\left(1 - e^{-0.1kT_s}\right)$$

and applying the Z-transform

$$Y^*(z) = 5\frac{z}{z-1} - 5\frac{z}{z-e^{-0.1T_s}}$$

Finally, the discrete time transfer function is given by

$$G^*(z) = Y^*(z)\frac{z-1}{z} = \left(5\frac{z}{z-1} - 5\frac{z}{z-e^{-0.1T_s}}\right)\frac{z-1}{z} = 5 - 5\frac{z-1}{z-e^{-0.1T_s}} = 5\frac{1-e^{-0.1T_s}}{z-e^{-0.1T_s}}$$