POLITECNICO DI MILANO





Automatic Control

State space design

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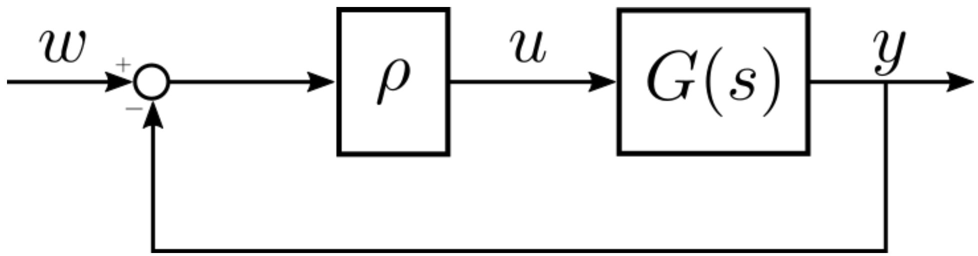
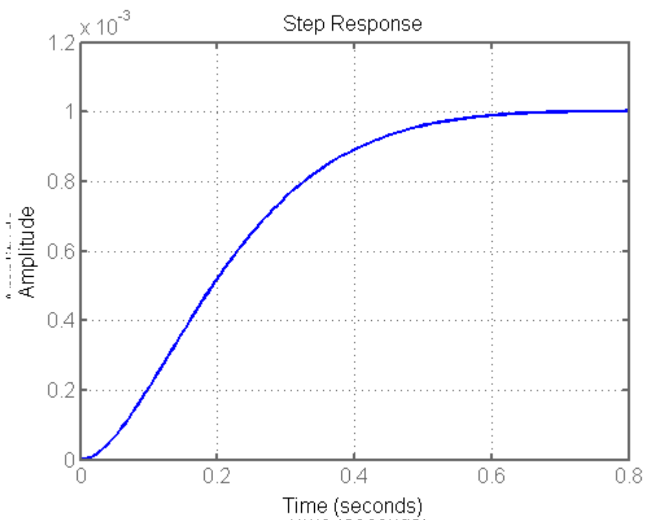
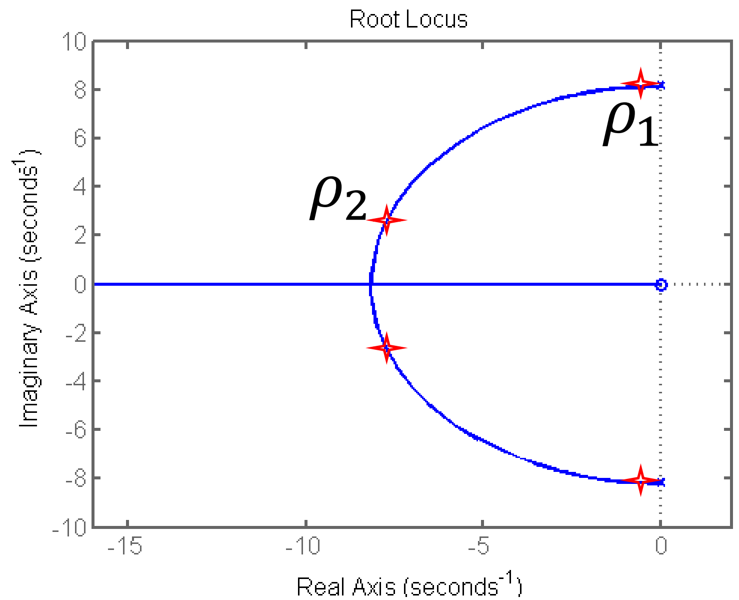
Motivations

Given the following closed-loop system



the root locus allows to shape the transient response, but…





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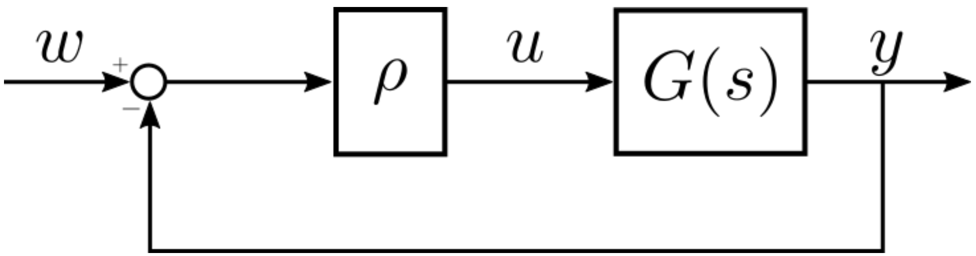

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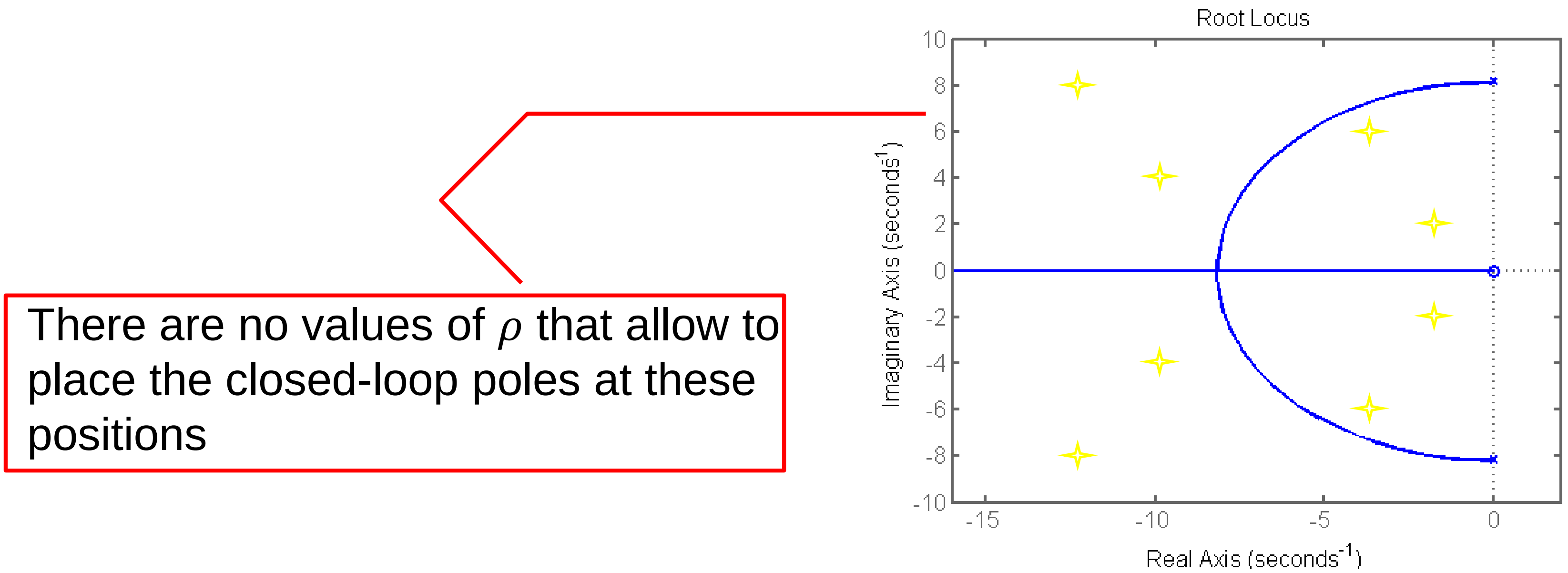


…the closed-loop poles cannot be placed at an arbitrary point of the complex plane.

Goals

Given a linear time invariant system, in the state-space formulation, pole placement allows

- to design a feedback control law

that

- places the closed-loop poles at arbitrary positions in the complex plane

Caveat: the desired positions may be either real numbers or complex numbers, with any complex root occurring in conjugate pairs!

As for root locus, pole placement can be exploited to

- stabilize an unstable system
- shape the transient response of a stable/unstable system

Full-State Feedback (I)

Consider a linear time invariant single-input plant, represented in state-space by

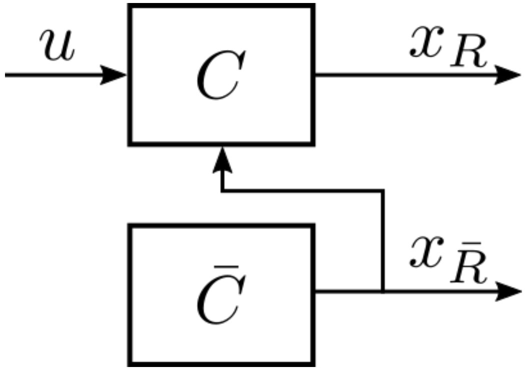
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

and an algebraic control law

$$u(t) = \mathbf{K}\mathbf{x}(t) \qquad \mathbf{K} = \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix}$$

that places the closed-loop poles at desired locations of the complex plane.

Caveat: remember that the input can only affect the controllable part of the system, as a consequence only poles of the controllable part can be arbitrarily placed.



We will assume that the plant is completely controllable.

Full-State Feedback (II)

Given the state-space

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

and the control law

$$u(t) = \mathbf{K}\mathbf{x}(t) \qquad \mathbf{K} = \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix}$$

the closed-loop system is described by the following state-space

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{B}\mathbf{K})\,\mathbf{x}(t)$$

Let's assume that the system is in controllable canonical form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\chi_{\mathbf{A}}(s) = \det(s\mathbf{I}_n - \mathbf{A}) = s^n + a_n s^{n-1} + \cdots + a_3 s^2 + a_2 s + a_1$$

Full-State Feedback (III)

The state matrix of the closed-loop system is given by

$$\mathbf{A}+\mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_{n-1} & k_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1+k_1 & -a_2+k_2 & -a_3+k_3 & \cdots & -a_n+k_n \end{bmatrix}$$

The state matrix of the closed-loop system is in controllable canonical form as well, and its characteristic polynomial is given by

$$\chi_{\mathbf{A}+\mathbf{BK}}(s) = s^n + (a_n - k_n)\, s^{n-1} + \cdots + (a_3 - k_3)\, s^2 + (a_2 - k_2)\, s + (a_1 - k_1)$$

Full-State Feedback (IV)

Decide now the closed-loop pole locations λ_i^o and determine an equivalent characteristic equation

$$\chi^o(s) = \prod_{i=1}^{n} (s - \lambda_i^o) = s^n + b_n s^{n-1} + \cdots + b_3 s^2 + b_2 s + b_1$$

If we equate like coefficients of the closed-loop characteristic equation

$$\chi_{\mathbf{A}+\mathbf{BK}}(s) = s^n + (a_n - k_n)\, s^{n-1} + \cdots + (a_3 - k_3)\, s^2 + (a_2 - k_2)\, s + (a_1 - k_1)$$

and the desired characteristic polynomial, and solve for k_i we obtain

$$k_i = a_i - b_i \quad i = 1, \ldots, n$$

In this case the control law that solves the pole placement exists and is unique.

Full-State Feedback (V)

What happens if the system is not in controllable canonical form?

Let's make a change of variables

$$\hat{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t)$$

that puts the system in controllable canonical form

$$\hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \quad \hat{\mathbf{B}} = \mathbf{T}\mathbf{B}$$

Solve the pole placement for the system in controllable canonical form, obtaining

$$u(t) = \hat{\mathbf{K}}\hat{\mathbf{x}}(t)$$

then, remembering the change of variable

$$u(t) = \hat{\mathbf{K}}\hat{\mathbf{x}}(t) = \hat{\mathbf{K}}\mathbf{T}\mathbf{x}(t) = \mathbf{K}\mathbf{x}(t)$$

In conclusion, the solution of the pole placement is given by

$$\mathbf{K} = \hat{\mathbf{K}}\mathbf{T}$$

Now the problem is, how to compute the matrix that defines the change of variables?

Full-State Feedback (VI)

Let's write the controllability matrix for the original system and for the system in controllable canonical form

$$\mathbf{K}_r = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \dots \end{bmatrix}$$

$$\hat{\mathbf{K}}_r = \begin{bmatrix} \hat{\mathbf{B}} & \hat{\mathbf{A}}\hat{\mathbf{B}} & \hat{\mathbf{A}}^2\hat{\mathbf{B}} & \dots \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{TB} & \mathbf{TAT}^{-1}\mathbf{TB} & \mathbf{TAT}^{-1}\mathbf{TAT}^{-1}\mathbf{TB} & \dots \end{bmatrix} = \mathbf{TK}_r$$

from these relations we obtain

$$\mathbf{T} = \hat{\mathbf{K}}_r \mathbf{K}_r^{-1}$$

We conclude that a <u>unique solution exists</u> to the full-state feedback problem, that allows to arbitrarily place at desired locations in the complex plane the closed-loop poles, if and only if the plant is completely controllable.

Full-State Feedback – An example (I)

Given the plant

$$\dot{x}_1 = -x_1 + 3x_2 + u$$

$$\dot{x}_2 = 2x_2 + 2u$$

$$y = x_1 + x_2$$

design a feedback law to place all the closed-loop poles at -1.

The state-space matrices are

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Let's first compute the controllability matrix

$$\mathbf{K}_r = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \qquad \det(\mathbf{K}_r) = -6 \neq 0$$

concluding that the system is completely controllable.

Full-State Feedback – An example (II)

The eigenvalues of the state matrix are -1 and 2, and the characteristic polynomial

$$\chi_{\mathbf{A}}(s) = (s+1)(s-2) = s^2 - s - 2$$

The desired characteristic polynomial, instead, is

$$\chi^{o}(s) = (s+1)(s+1) = s^2 + 2s + 1$$

Equating like coefficients of the two characteristic equations yields

$$\begin{aligned} \hat{k}_1 &= a_1 - b_1 = -2 - 1 = -3 \\ \hat{k}_2 &= a_2 - b_2 = -1 - 2 = -3 \end{aligned} \quad \Rightarrow \quad \hat{\mathbf{K}} = \begin{bmatrix} -3 & -3 \end{bmatrix}$$

that is the gain of the control law that places the closed-loop poles of the system in controllable canonical form.

The original system, however, is not in controllable canonical form, we thus need to compute the transformation matrix.

Remembering that the characteristic polynomial is

$$\chi_{\mathbf{A}}(s) = (s+1)(s-2) = s^2 - s - 2$$

The controllable canonical form is

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \qquad \hat{\mathbf{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

whose controllability matrix is given by

$$\hat{\mathbf{K}}_r = \begin{bmatrix} \hat{\mathbf{B}} & \hat{\mathbf{A}}\hat{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The transformation matrix is thus

$$\mathbf{T} = \hat{\mathbf{K}}_r \mathbf{K}_r^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -4 & 5 \\ 2 & -1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}$$

and the control law for the original plant is given by

$$\mathbf{K} = \hat{\mathbf{K}}\mathbf{T} = \begin{bmatrix} -3 & -3 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2} \end{bmatrix}$$

Full-State Feedback for multi-input plants

What happens if the plant has $m > 1$ inputs?

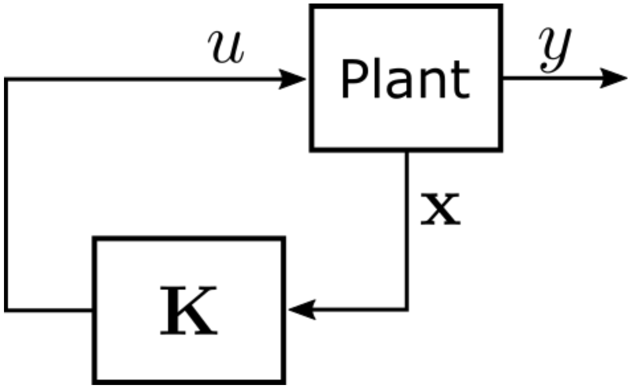
In this case the input matrix is $n \times m$ and we can experience two different situations

- there is at least one (or there are more) input that makes the system completely controllable (i.e., $\exists i \in [1, m]: (\mathbf{A}, \mathbf{B}_i)$ is completely controllable)
  - for each input that makes the system completely controllable, one can solve a single-input full-state feedback problem (i.e., determining a gain matrix $\mathbf{K}_i$ that places the poles of $\mathbf{A} + \mathbf{B}_i\mathbf{K}_i$ at desired locations in the complex plane)
- $\nexists i \in [1, m]: (\mathbf{A}, \mathbf{B}_i)$ is completely controllable, but the multi-input system $(\mathbf{A}, \mathbf{B})$ is completely controllable
  - at least one $m \times n$ matrix $\mathbf{K}$ exists that places the poles of $\mathbf{A} + \mathbf{BK}$ at desired locations in the complex plane

We conclude that in the case of multi-input plants the pole placement problem can have <u>multiple solutions</u>.

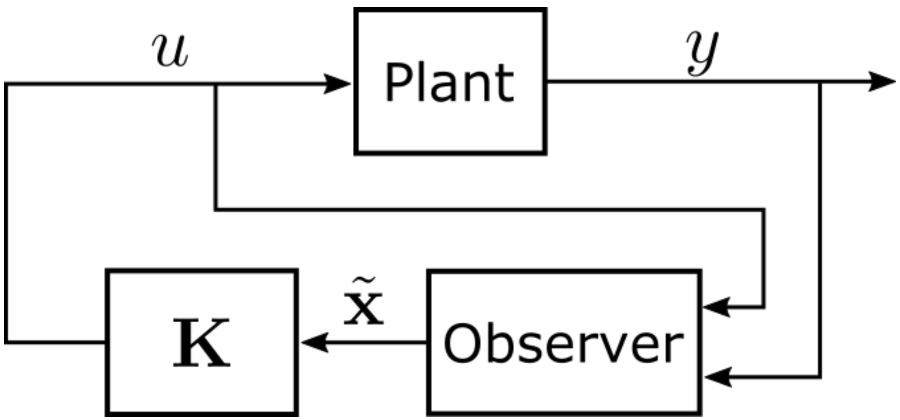
Observer design (I)

The full-state feedback assumes that all the plant state variables are available for feedback.



In most cases, not all the state variables are measurable/measured.

We thus need a tool to reconstruct the plant state variables from a set of measurements.



Observer design (II)

Consider a linear time invariant, strictly proper and SISO plant represented in state-space by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$\mathbf{x}(0) = \mathbf{x}_0$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

Let's assume that the state-space matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are known without uncertainty.

We will not enforce any assumption on the stability of the plant.

Let's consider a copy of the original plant

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}u(t)$$
$$\tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0$$
$$\tilde{y}(t) = \mathbf{C}\tilde{\mathbf{x}}(t)$$

Observer design (III)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$\mathbf{x}(0) = \mathbf{x}_0$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}u(t)$$
$$\tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0$$
$$\tilde{y}(t) = \mathbf{C}\tilde{\mathbf{x}}(t)$$

Under the previous assumptions, if the plant and its copy are fed by the same input and start from the same initial state, they will produce the same output.

If the initial state is unknown or known with uncertainty, the two outputs will be different. To solve this issue we introduce a corrective term that depends on the difference between the two outputs

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\left(\tilde{y}(t) - y(t)\right)$$
$$\tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0$$
$$\tilde{y}(t) = \mathbf{C}\tilde{\mathbf{x}}(t)$$

where $\mathbf{L}$ ($n \times 1$) is a vector of weights.

What happens in a real problem?

The plant is a physical system (whose model is represented by a LTI SISO dynamical system) with an input variable u and a measurement y.

The observer is an LTI SISO dynamical system (i.e., an algorithm), fed by the plant input and measurement, whose aim is to estimate the plant state variables.

The goal of the observer is to make the state estimation error

$$\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t)$$

vanishing, at least asymptotically.

How can we describe the behavior of the state estimation error?

Let's consider the time derivative of the error

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\tilde{\mathbf{x}}}(t) \\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) - \mathbf{A}\tilde{\mathbf{x}}(t) - \mathbf{B}u(t) - \mathbf{LC}\left(\tilde{\mathbf{x}}(t) - \mathbf{x}(t)\right) \\ &= \left(\mathbf{A} + \mathbf{LC}\right)\mathbf{e}(t)\end{aligned}$$

Observer design (V)

We conclude that the state estimation error is the solution of a LTI autonomous system

$$\dot{\mathbf{e}}(t) = (\mathbf{A} + \mathbf{LC})\,\mathbf{e}(t)$$

if we can select $\mathbf{L}$ to arbitrarily place the poles of $\mathbf{A} + \mathbf{LC}$, we can guarantee that the estimation error asymptotically vanishes with a desired transient response.

Is this the same problem as placing the poles of $\mathbf{A} + \mathbf{BK}$?

$$\lambda_i\left[\mathbf{A} + \mathbf{LC}\right] = \lambda_i\left[(\mathbf{A} + \mathbf{LC})^T\right] = \lambda_i\left[\mathbf{A}^T + \mathbf{C}^T\mathbf{L}^T\right]$$

Yes, it is the same problem, thus

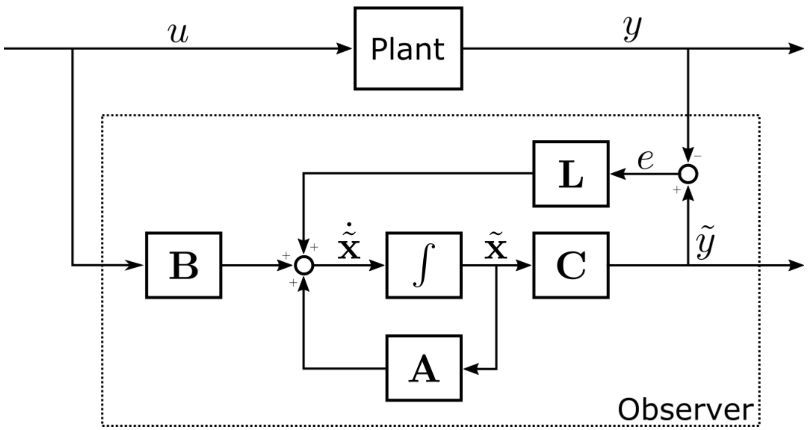
- if $(\mathbf{A}^T, \mathbf{C}^T)$ is completely controllable (i.e., $(\mathbf{A}^T, \mathbf{C}^T)$ is completely controllable if and only if $(\mathbf{A}, \mathbf{C})$ is completely observable)
- we can solve the pole placement problem with $\mathbf{K} = \mathbf{L}^T$

Summarizing the pole placement procedure, to compute $\mathbf{L}^T$ (assuming $(\mathbf{A}, \mathbf{C})$ is completely observable) we

- compute $\mathbf{K}_o$, $\widehat{\mathbf{K}}_o$ and $$\mathbf{T} = \hat{\mathbf{K}}_o \mathbf{K}_o^{-1}$$
- compute $\hat{\mathbf{L}}^T$ solving pole placement for the system $(\widehat{\mathbf{A}}^T, \widehat{\mathbf{C}}^T)$ in controllable canonical form
- compute the observer gain

$$\mathbf{L}^T = \hat{\mathbf{L}}^T \mathbf{T} \quad \Rightarrow \quad \mathbf{L} = \mathbf{T}^T \hat{\mathbf{L}}$$

We conclude that a unique solution $\mathbf{L}$ exists, that allows to arbitrarily place at desired locations in the complex plane the poles of the state estimation error dynamics, if and only if the plant is completely observable.



Observer design (VII)

We conclude with some remarks:

- with multi-output systems the solution of the observer design problem is not unique (as it happens with full-state feedback)
- in most cases not all the state variables need to be reconstructed (i.e., some of them can be directly measured) and a reduced order observer (i.e., an observer whose order is less then the order of the plant) is more appropriate

The dual system

Given a linear time invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

we define dual system the following dynamical system

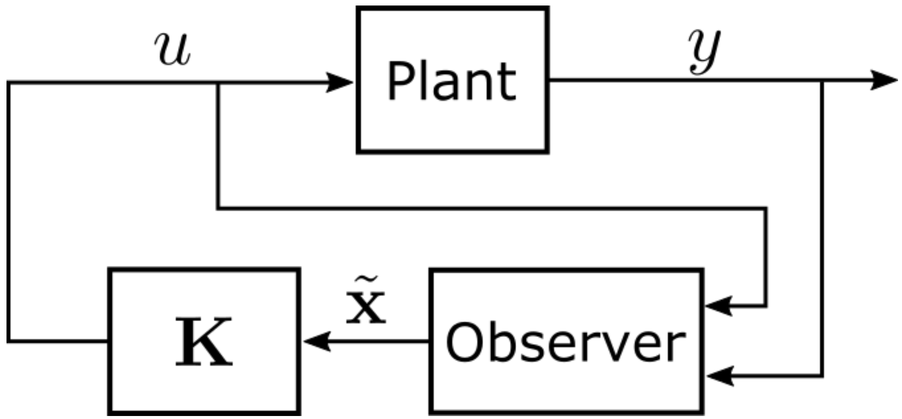
$$\dot{\mathbf{x}}(t) = \mathbf{A}^T\mathbf{x}(t) + \mathbf{C}^T\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{B}^T\mathbf{x}(t)$$

having the following properties

- the dual system is asymptotically stable if and only if the original system is asymptotically stable (i.e., $\lambda_i(\mathbf{A}) = \lambda_i(\mathbf{A}^T)$)
- the dual system is completely controllable if and only the original system is completely observable
- the dual system is completely observable if and only if the original system is completely controllable

It’s now time to study the combination of the observer and the pole placement control law.



We will now assume that the plant is

- completely controllable
- completely observable

Combining observer and control law (II)

Let's consider the equations of the plant, the observer and the pole placement law

Plant

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

Observer

$$\begin{cases} \dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\left(\tilde{y}(t) - y(t)\right) \\ \tilde{y}(t) = \mathbf{C}\tilde{\mathbf{x}}(t) \end{cases}$$

Pole placement law

$$u(t) = \mathbf{K}\tilde{\mathbf{x}}(t)$$

Merging the equations we obtain

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{K}\tilde{\mathbf{x}}(t)$$

$$\dot{\tilde{\mathbf{x}}}(t) = -\mathbf{L}\mathbf{C}\mathbf{x}(t) + \left(\mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{C}\right)\tilde{\mathbf{x}}(t)$$

Let's now do a change of variables, introducing the state estimation error

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} + \mathbf{B}\mathbf{K}\right)\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{e}(t)$$

$$\dot{\mathbf{e}}(t) = \left(\mathbf{A} + \mathbf{L}\mathbf{C}\right)\mathbf{e}(t)$$

Combining observer and control law (III)

The closed-loop system is thus represented by the following equation in matrix form

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A}+\mathbf{BK} & -\mathbf{BK} \\ \hline \mathbf{0} & \mathbf{A}+\mathbf{LC} \end{array}\right] \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix}$$

an autonomous system with a block triangular state matrix.

The closed-loop poles are the union of the eigenvalues of matrices $\mathbf{A}+\mathbf{BK}$ and $\mathbf{A}+\mathbf{LC}$.

If the plant is completely controllable and completely observable we can arbitrarily place the poles of $\mathbf{A}+\mathbf{BK}$ and $\mathbf{A}+\mathbf{LC}$.

Combining observer and control law (IV)

We conclude the analysis of the closed-loop system with some remarks:

- if n is the order of the plant, the closed-loop system has order $2n$
- the pole placement control law and the observer gain can be designed independently. In particular, the pole placement law can be designed as if all the state-variables were measurable; the observer gain as if there were no feedback (separation principle)
- though, from a theoretical point of view, the closed-loop poles can be arbitrarily place in any position of the complex plane, there are obvious practical reasons (as in any closed-loop system) that prevent from increasing too much the speed of the closed-loop system with respect to the open loop one (“reasonably small” control efforts)

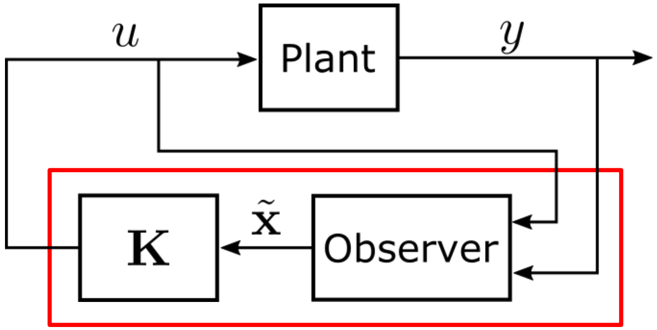
Combining observer and control law (V)

Merging the equation of the observer with the control law we obtain the equations of the compensator

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\left(\tilde{y}(t) - y(t)\right)$$
$$\tilde{y}(t) = \mathbf{C}\tilde{\mathbf{x}}(t)$$
$$u(t) = \mathbf{K}\tilde{\mathbf{x}}(t)$$

simplifying these equations, we obtain the expression of the compensator

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A} + \mathbf{BK} + \mathbf{LC})\,\tilde{\mathbf{x}}(t) - \mathbf{L}y(t)$$
$$u(t) = \mathbf{K}\tilde{\mathbf{x}}(t)$$



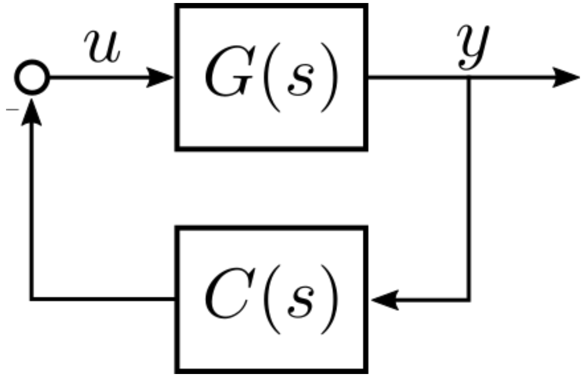
Caveat: the eigenvalues of matrix $\mathbf{A} + \mathbf{BK} + \mathbf{LC}$ are not directly related to the eigenvalues of $\mathbf{A} + \mathbf{BK}$ or $\mathbf{A} + \mathbf{LC}$, there is thus no guarantee that the compensator is asymptotically stable.

Finally, defining the transfer functions of the plant and the compensator

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) \end{cases} \Rightarrow \quad G(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}$$

$$\begin{cases} \dot{\tilde{\mathbf{x}}}(t) = \left(\mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{C}\right)\tilde{\mathbf{x}}(t) - \mathbf{L}y(t) \\ u(t) = \mathbf{K}\tilde{\mathbf{x}}(t) \end{cases} \Rightarrow \quad C(s) = \mathbf{K}\left(s\mathbf{I} - \left(\mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{C}\right)\right)^{-1}\mathbf{L}$$

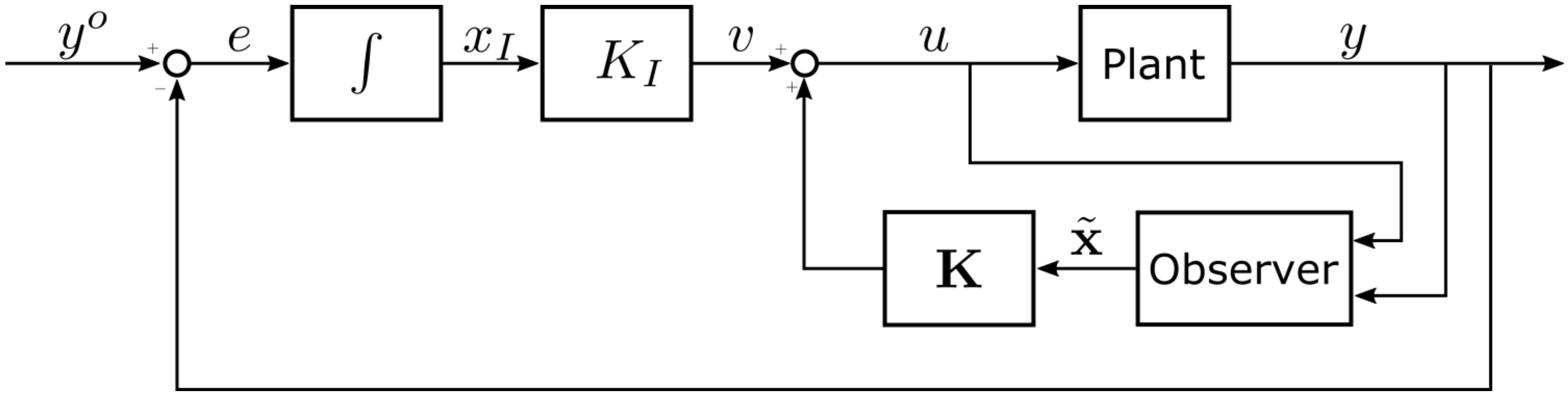
we clearly see that they are connected in a standard negative feedback loop.



Introduction of a reference input (I)

Up to now pole placement has been used to stabilize the closed-loop system and/or to shape the transient response, but no reference input has been considered.

If we now consider a reference tracking problem, we have to ensure that the tracking error asymptotically vanishes. We thus need to introduce an integrator.



We can design the gain K_I together with the matrix gain $\mathbf{K}$ solving a pole placement problem on the augmented system including the plant and the state of the integrator.

The augmented state equations become

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$\dot{x}_I(t) = y^o(t) - y(t) = y^o(t) - \mathbf{C}\mathbf{x}(t)$$

or, in matrix form

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{x}_I(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} \mathbf{x}(t) \\ x_I(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}}_{\mathbf{G}_u} u(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{G}_{y^o}} y^o(t)$$

If $(\mathbf{F}, \mathbf{G}_u)$ is completely controllable, we can arbitrarily assign the poles of the augmented system.

Caveat: it can be shown that the pole placement problem can be solved if and only if the plant does not have zeros in $s = 0$.

Let's analyze the previous condition.

The system controllability matrix is

$$\mathbf{K}_{r_{(\mathbf{F},\mathbf{G}_u)}} = \begin{bmatrix}\mathbf{G}_u & \mathbf{F}\mathbf{G}_u & \mathbf{F}^2\mathbf{G}_u & \dots & \mathbf{F}^n\mathbf{G}_u\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^n\mathbf{B} \\ 0 & -\mathbf{C}\mathbf{B} & -\mathbf{C}\mathbf{A}\mathbf{B} & \dots & -\mathbf{C}\mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}0 & \mathbf{B} & \mathbf{A}\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \\ 1 & 0 & 0 & \dots & 0\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}0 & \mathbf{K}_r \\ 1 & \mathbf{0}\end{bmatrix}$$

The second matrix is non singular as the plant is completely controllable.

What about the first matrix?

Introduction of a reference input (IV)

If $\mathbf{A}$ is non singular the determinant of the first matrix can be computed as (Schur complement rule)

$$\det\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ -\mathbf{C} & 0\end{bmatrix} = \det(\mathbf{A})\det\left(\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)$$

The first matrix is thus non singular if and only if

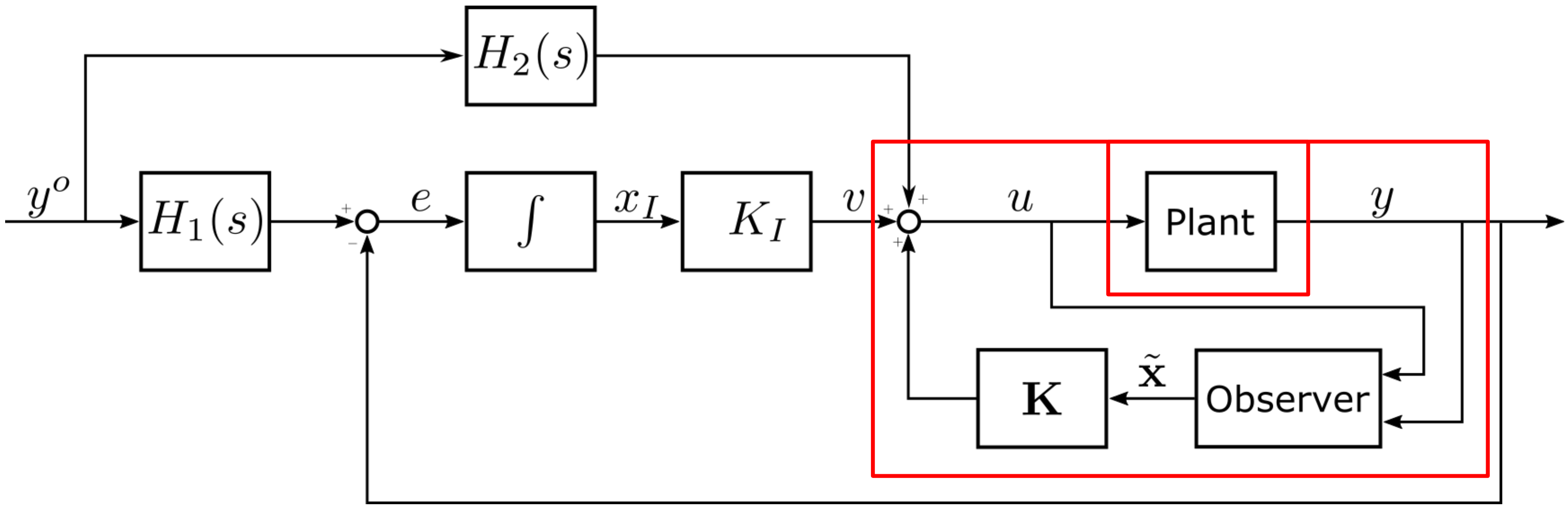
$$\det\left(\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right) = \det(-G(0)) = -G(0) \neq 0$$

and the transfer function is non singular if and only if the system does not have zeros in $s = 0$.

Note that this property holds even if $\mathbf{A}$ is singular.

Adding a feedforward action (I)

The transient response of the closed-loop system can be further improved adding a feedforward action.

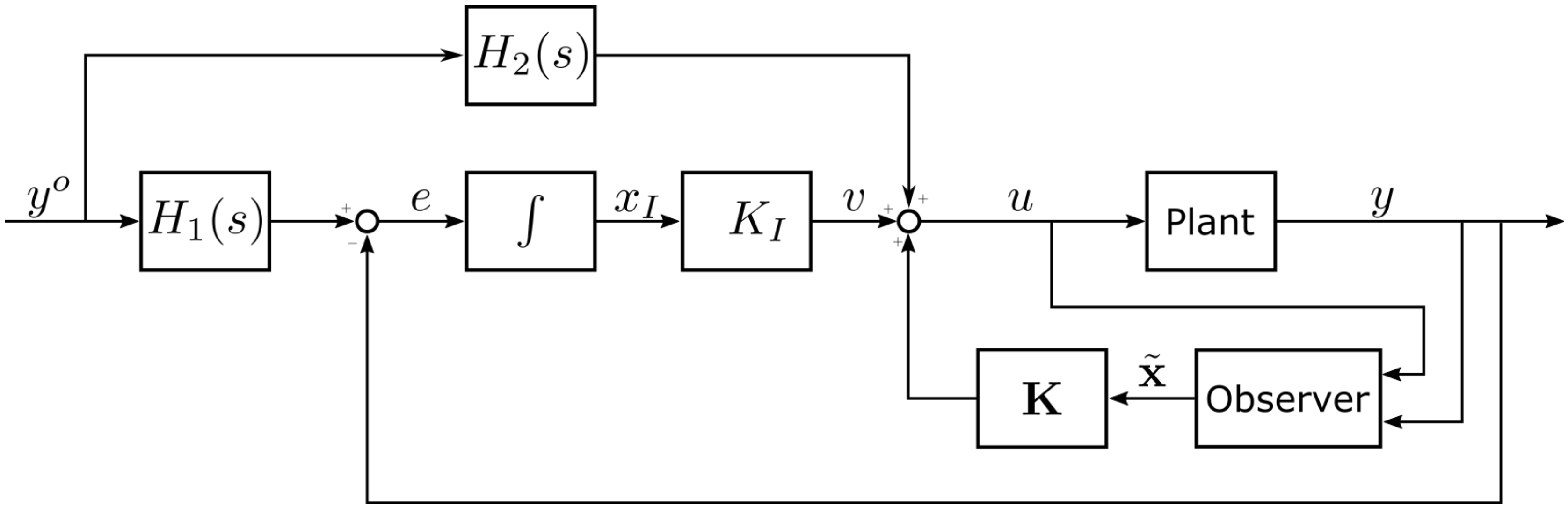


Define the following transfer functions

$$\frac{Y(s)}{U(s)} = G(s) = \frac{N(s)}{D(s)}$$

$$\frac{Y(s)}{V(s)} = G_k(s) = \frac{N(s)}{D_k(s)} = \frac{N(s)}{\det\left(s\mathbf{I} - (\mathbf{A} + \mathbf{BK})\right)}$$

Adding a feedforward action (II)



The transfer function from the reference to the controlled variable is

$$\frac{Y(s)}{Y^o(s)} = \frac{\left[\frac{H_1(s)K_I}{s} + H_2(s)\right] G_k(s)}{1 + \frac{K_I}{s} G_k(s)}$$

The feedforward is designed in order to have the closed-loop system behaving like a reference model

$$\frac{Y(s)}{Y^o(s)} = F^o(s) \quad \Rightarrow \quad H_1(s) = F^o(s) \quad H_2(s) = F^o(s) G_k(s)^{-1}$$

Adding a feedforward action (III)

Caveat: to ensure the realizability and stability of the closed-loop system, the reference model must:

- be a unitary gain transfer function
- have a relative degree no less than $G_K(s)$ relative degree
- contain as zeros all the zeros of $G_K(s)$ that lies in the right half plane