Control of Mobile Robots

Kinematics of mobile robots

Prof. Luca Bascetta (luca.bascetta@polimi.it)

Politecnico di Milano – Dipartimento di Elettronica, Informazione e Bioingegneria

Organization and motivations

Before studying the motion control problem for a mobile robot, we have to introduce the modeling tools

- to describe the instantaneous admissible motions of the robot (kinematic model)
- to relate these motions to the generalized forces acting on the robot (dynamic model)

The main topics on kinematic modelling are

- kinematics review
- constraints
- using constraints to derive the kinematic model of a mechanical system
- deriving kinematic models of mobile robots
- a system theory interpretation of holonomy and nonholonomy

Organization and motivations

We focus on kinematic constraints to introduce a modelling tool that allows to derive the kinematic model of any mobile robot, always applying the same general procedure.

Consider the following examples:

- a car
- a quadrotor
- an underwater spherical vehicle

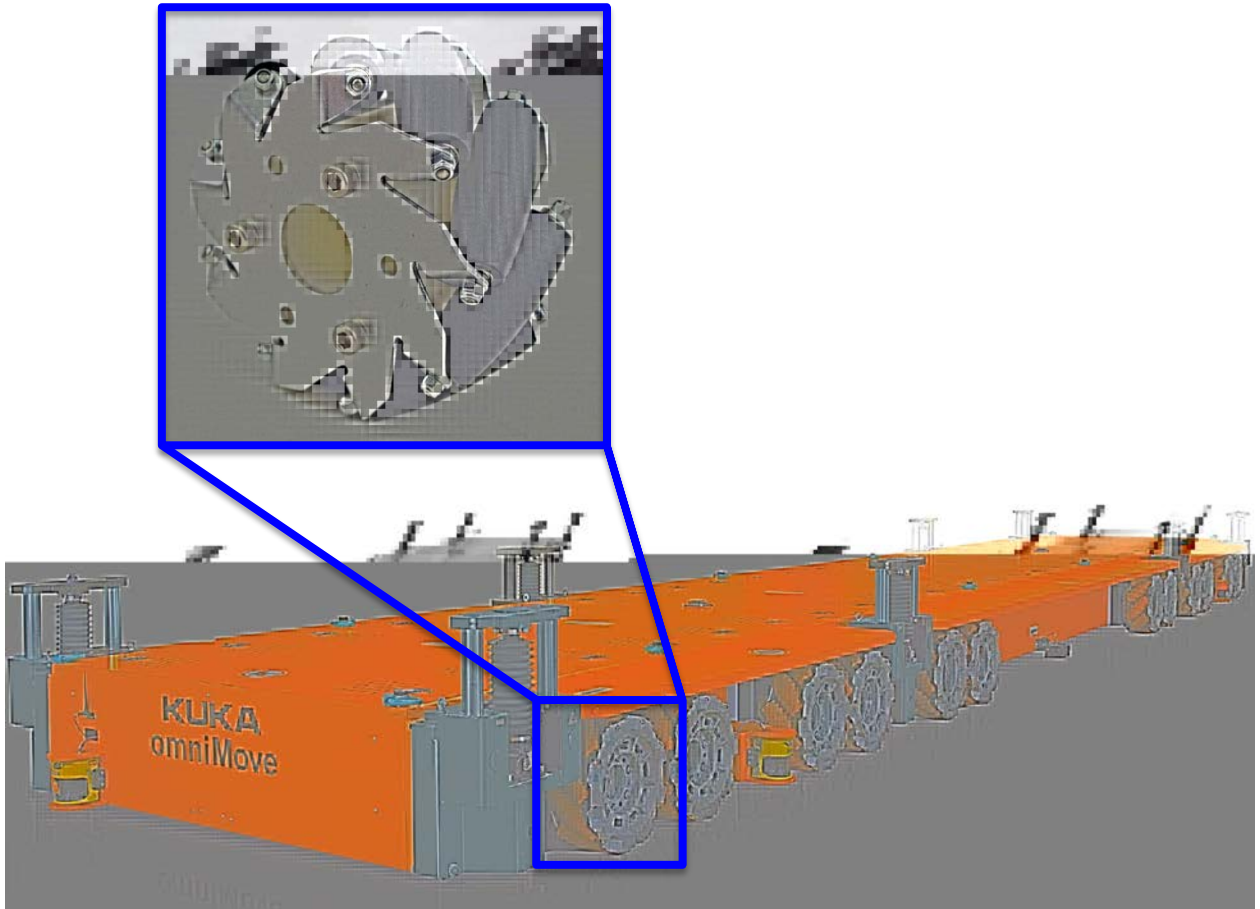
for each of them we would like to derive the kinematic model…

Cars are characterised by a steering mechanism, a constraint that reduces its local mobility

Vehicles with double steering mechanisms exist as well!

Wheels, locomotion and constraints

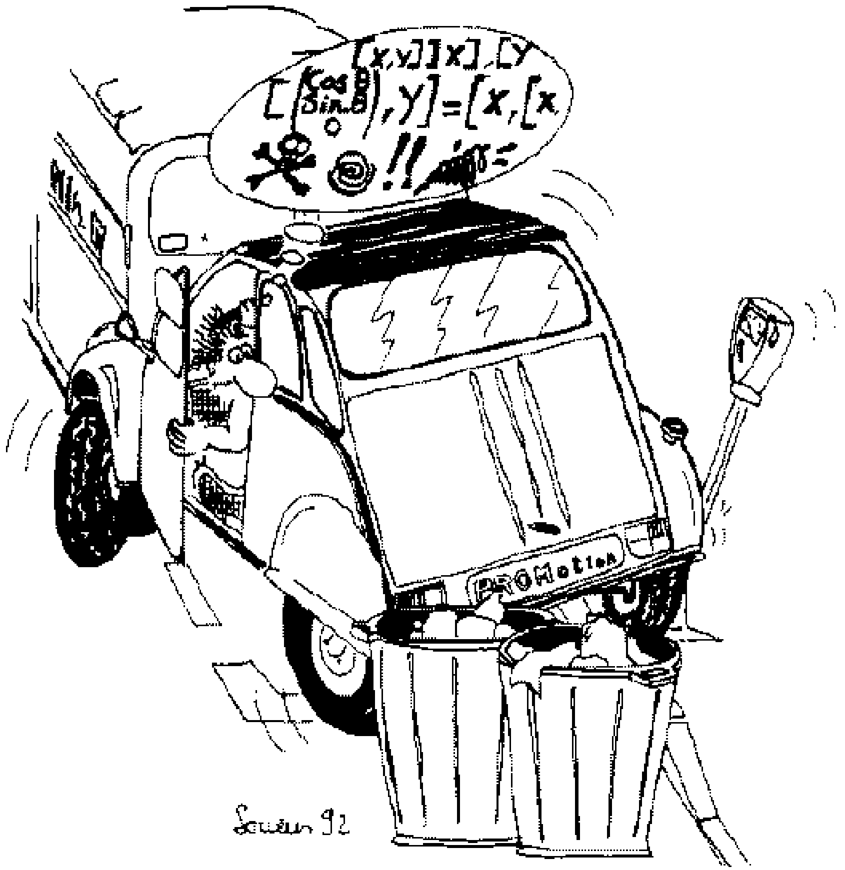
Other vehicles turn using skidding



Others can move in any direction without constraints

Constraints and locomotion

- A hinge introduces a constraint that reduces the mobility of the pendulum
- A body moving on a plane has 3 d.o.f., the pendulum instead is characterized by only 1 d.o.f.
- Steering mechanism reduces the mobility of a car in a completely different way
- A car moves on a plane, but despite the constraint it still has 3 d.o.f.



Consider a particle P in a 3D space, whose position with respect to a reference frame $O - xyz$ is defined by the vector $\boldsymbol{P} - \boldsymbol{O}$.

The motion of a particle is ***free*** if it is not subjected to any constraint, otherwise it is ***constrained*** (e.g., it must lie on a surface, on a line, …).

Every constraint is represented by an analytic relation between the coordinates xyz.

If the particle must lie on a surface, the coordinates should satisfy the equation

$$f(x, y, z) = 0$$

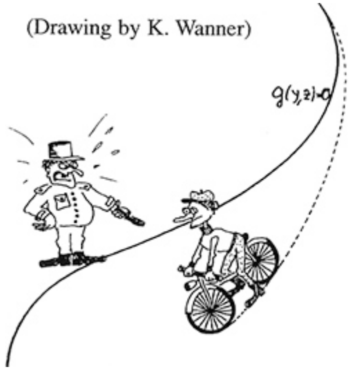
If it must lie on a line, the coordinates should satisfy the equations

$$\varphi(x, y, z) = 0 \qquad \psi(x, y, z) = 0$$

If it cannot cross a surface, the coordinates should satisfy the inequality

$$f(x, y, z) \leq 0$$

(Drawing by K. Wanner)



Constraints can be classified as:

- *bilateral constraints*, described by equality constraints
- *unilateral constraints*, described by inequality constraints

When a particle is subjected to bilateral constraints its position in the 3D space is represented by less then 3 coordinates

- if it is constrained to move on a line, its position is described by 1 coordinate
- if it is constrained to move on a surface, its position is described by 2 coordinates

A system of particles is a (finite or infinite) set of free or constrained particles.

Consider a system of N particles P_i in a 3D space, each one characterized by a position with respect to a reference frame $O - xyz$ defined by the vector $\boldsymbol{P}_i - \boldsymbol{O}$, $i = 1, 2, \dots, N$.

Let's introduce a *rigidity constraint*, i.e., the distance between any two particles is always constant.

Thanks to this constraint, we can use less than $3N$ coordinates to define the *system configuration* (position).

Let's consider an example.

Consider a system of 2 particles, $\boldsymbol{P}_1(x_1, y_1, z_1)$ and $\boldsymbol{P}_2(x_2, y_2, z_2)$, moving in the xy plane and subjected to a rigidity constraint.

The 6 coordinates describing the position of the 2 particles are related by the following constraints

$$\begin{cases} (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = c^2 \\ z_1 = 0 \\ z_2 = 0 \end{cases}$$

We can thus reduce the coordinates to 4, neglecting z_1 and z_2, and introducing the constraint

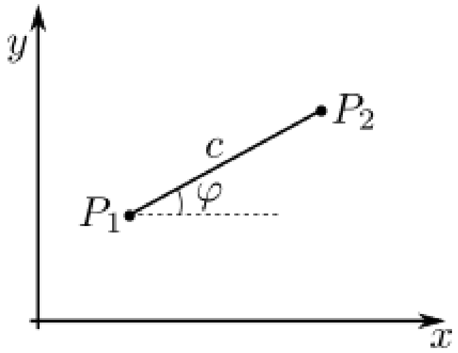
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = c^2$$

But these 4 coordinates are not independent, i.e., they can be further reduced!

The configuration of the system of 2 particles can be described by 3 independent coordinates.

For example x_1, y_1, φ or x_2, y_2, φ.

We can select different sets of independent coordinates, but the cardinality of these sets is always 3.

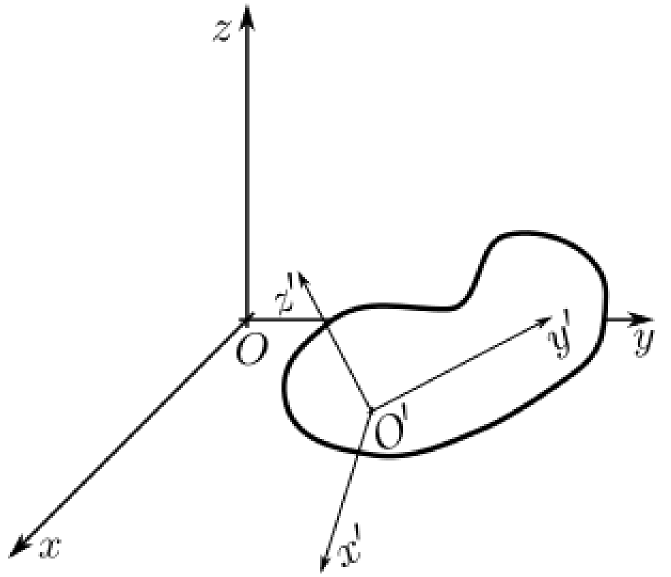


The number of independent coordinates required to represent the configuration of a system of particles determines the number of *degrees of freedom* (DOF) of the system.

A system of N particles subjected to r constraints is thus described by $3N - r$ *Lagrange or generalized coordinates*, and it is thus characterized by $3N - r$ DOF.

Consider a rigid body and

- a fixed reference frame $O - xyz$
- a moving reference frame $O' - x'y'z'$



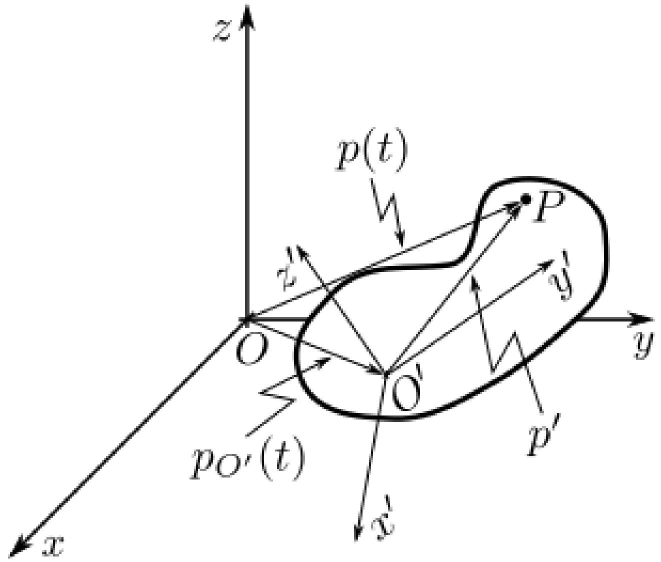
The orientation of the moving frame with respect to the fixed frame can be expressed with an orthonormal rotation matrix

$$R(t) = \begin{bmatrix} x'^T(t)x & y'^T(t)x & z'^T(t)x \\ x'^T(t)y & y'^T(t)y & z'^T(t)y \\ x'^T(t)z & y'^T(t)z & z'^T(t)z \end{bmatrix}$$

The columns represent the components of the unit vectors of the moving frame expressed in the fixed frame.

Let $\boldsymbol{p}'$ be the constant position of a generic point $\boldsymbol{P}$ in the moving frame, the motion of $\boldsymbol{P}$ with respect to the fixed frame $O - xyz$ is described by

$$\mathbf{p}(t) = \mathbf{p}_{O'}(t) + R(t)\mathbf{p}'$$



The position and orientation (*pose*) of a rigid body is thus represented by:

- a position vector (3 independent parameters)
- a rotation matrix (9 parameters and 6 constraints = 3 independent parameters)

The motion of a rigid body is thus described by 6 independent functions of time (6 DOF).

Differentiating with respect to time the expression

$$\mathbf{p}(t) = \mathbf{p}_{O'}(t) + R(t)\mathbf{p}'$$

gives the velocity of the generic point P

$$\dot{\mathbf{p}} = \dot{\mathbf{p}}_{O'} + \dot{R}\mathbf{p}' = \dot{\mathbf{p}}_{O'} + S(\omega)R\mathbf{p}' = \dot{\mathbf{p}}_{O'} + \omega \times R\mathbf{p}'$$

and differentiation again with respect to time we obtain the accelerations

$$\ddot{\mathbf{p}} = \ddot{\mathbf{p}}_{O'} + \dot{\omega} \times R\mathbf{p}' + \omega \times \left(\omega \times R\mathbf{p}'\right)$$

How is matrix $S(\omega)$ defined?

What are the relations among $R(t)$, $S(\omega)$ and ω?

What are the properties of matrix $S(\omega)$?

From the orthogonality of a rotation matrix it follows that

$$R(t)R^T(t) = I$$

differentiation with respect to time gives

$$\dot{R}(t)R^T(t) + R(t)\dot{R}^T(t) = 0$$

We define matrix $S(\omega)$ as

$$S(t) = \dot{R}(t)R^T(t)$$

From the previous relation it follows that (*skew-symmetric property*)

$$S(t) + S^T(t) = 0$$

Finally the derivative of a rotation matrix is given by

$$\dot{R}(t) = S(t)R(t)$$

Consider a constant vector $\boldsymbol{p}'$ and the vector

$$\mathbf{p}(t) = R(t)\,\mathbf{p}'$$

The time derivative of $\boldsymbol{p}(t)$ is

$$\dot{\mathbf{p}}(t) = \dot{R}(t)\,\mathbf{p}'$$

which can be rewritten as

$$\dot{\mathbf{p}}(t) = S(t)\,R(t)\,\mathbf{p}'$$

If vector $\boldsymbol{\omega}(t) = [\omega_x \quad \omega_y \quad \omega_z]$ denotes the angular velocity of frame $R(t)$ with respect to a fixed reference frame at time t, it can be shown that

$$\dot{\mathbf{p}}(t) = \omega(t) \times R(t)\,\mathbf{p}'$$

and $S(\boldsymbol{\omega})$ can be written as

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Kinematics review – Derivative of a rotation matrix

Finally, if R is a rotation matrix it can be shown that

$$RS(\omega)R^T = S(R\omega)$$

…and an $m \times m$ rotation matrix R belongs to the *special orthonormal group* $SO(m)$ of the real $m \times m$ matrices with orthonormal columns and determinant equal to 1.

Let's now come back to kinematic constraints…

Constraints

We have already introduced a classification of constraints, and the preliminary examples show that:

- there are constraints reducing the number of DOF of the system…
- …and other constraints reducing only the mobility of the system.

Let's try to introduce a more accurate classification:

- *bilateral* / *unilateral* ⇒ equality / inequality constraints
- *rheonomic* / *scleronomic* ⇒ time dependent / time independent constraints
- *holonomic* / *nonholonomic*

Let's now analyze holonomic and nonholonomic constraints, making reference to bilateral scleronomic constraints…

Holonomic constraints

Consider a mechanical system whose configuration is described by $\boldsymbol{q} \in \mathcal{C} \equiv \mathbb{R}^n$.

A constraint is called *holonomic* (or *integrable*) if it can be written in the following form

$$h_i(\mathbf{q}) = 0 \qquad i = 1, \ldots, k < n$$

These constraints reduce the configuration space to a subset of $\mathcal{C}$ with dimension $n - k$.

Assuming that

- h_i are functions of class C^∞
- Jacobian $\partial \boldsymbol{h} / \partial \boldsymbol{q}$ has maximum rank

the implicit function theorem allows to (locally) solve the constraints expressing k generalized coordinates as a function of the remaining $n - k$.

Consequently, a reduced set of $n - k$ generalized coordinates, describing only the available DOF, can be introduced.

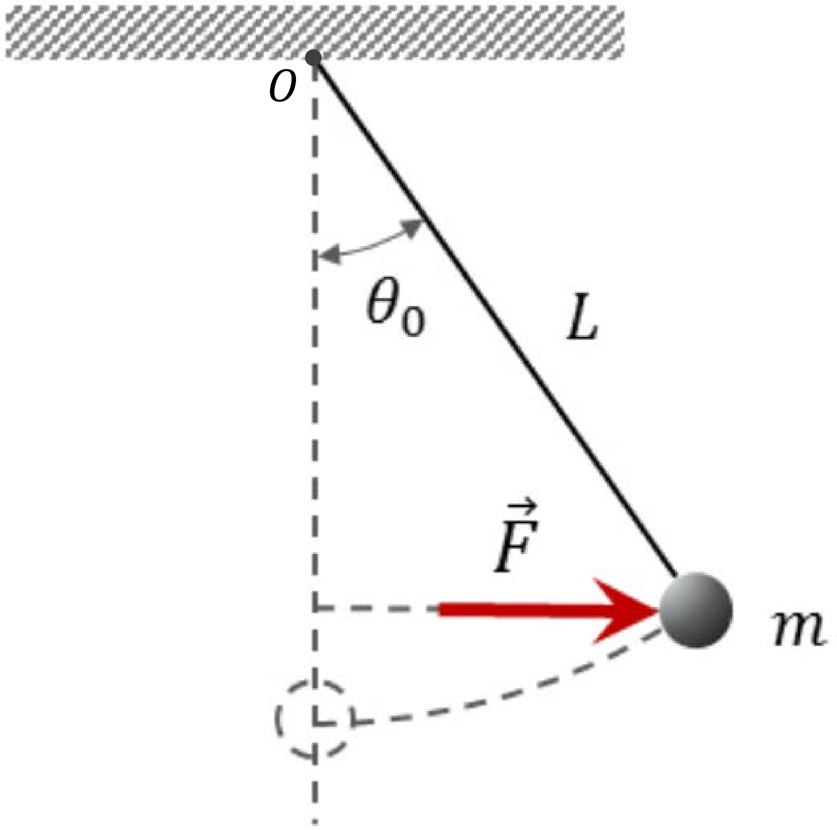
Determine the number of DOF and the set of generalized coordinates of a pendulum

Consider a pendulum (single link robot), the mass m is constrained to move in the vertical plane.

- A rigid body in the 3D space: 6 DOF $(x, y, z, \phi, \theta, \psi)$
- The planar motion constraint: 3 DOF (x, y, θ)
- The hinge introduces two further constraints

$$x_O = 0 \qquad y_O = 0$$

- The pendulum has thus 3-2 = 1 DOF
- A new set of generalized coordinates able to describe the available degrees of freedom is represented by angle θ



Kinematic constraints

What happens differentiating the holonomic constraints $h_i(\boldsymbol{q}) = 0$ with respect to time?

$$\frac{\mathrm{d}h_i(\mathbf{q})}{\mathrm{d}t} = \frac{\partial h_i(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}} \qquad i = 1, \dots, k$$

This new set of k constraints can be rewritten as

$$\mathbf{a}_i^T(\mathbf{q})\,\dot{\mathbf{q}} = 0 \qquad i = 1, \dots, k$$

or, in matrix form, as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \mathbf{0}$$

This is the most common form, called *Pfaffian form*, in which *kinematic constraints* are expressed, i.e., linear with respect to generalized velocities.

Kinematic constraints

Kinematic constraints limit the instantaneous admissible motion of the system by reducing the set of generalized velocities that can be attained at each configuration.

They can be expressed in a more general form as

$$a_i(\mathbf{q}, \dot{\mathbf{q}}) = 0 \qquad i = 1, \ldots, k$$

We can conclude that the existence of k holonomic constraints implies the existence of k kinematic constraints.

Is the converse true in general?

Consider a single Pfaffian constraint

$$a^T(\mathbf{q})\dot{\mathbf{q}} = 0$$

If a holonomic constraint associated to this kinematic constraint exists, we should be able to integrate the Pfaffian constraint to obtain

$$h(\mathbf{q}) = c$$

If, however, the kinematic constraint is non-integrable, we will call it *nonholonomic constraint*.

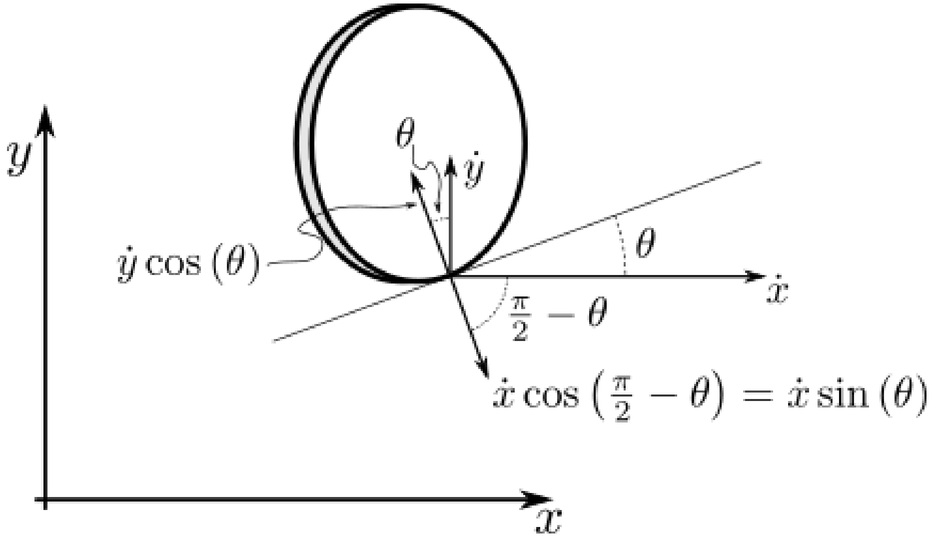
How can we interpret constraint $a^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0$?

The generalized velocities are constrained to belong to $\mathrm{Null}\big(a^T(\boldsymbol{q})\big)$.

But there is no constraint, neither locally, that allows to decrease the number of generalized coordinates, i.e., there is no loss of accessibility for the system.

Determine the nonholonomic constraint characterising the motion of a rolling disk

Consider a disk rolling without slipping on the horizontal plane, keeping the sagittal plane in the vertical direction.



The disk configuration is described by 3 generalized coordinates $\boldsymbol{q} = [x \quad y \quad \theta]^T$.

In order to satisfy the pure rolling constraint, we must guarantee that the velocity of the contact point has zero component in the direction orthogonal to the sagittal plane

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$$

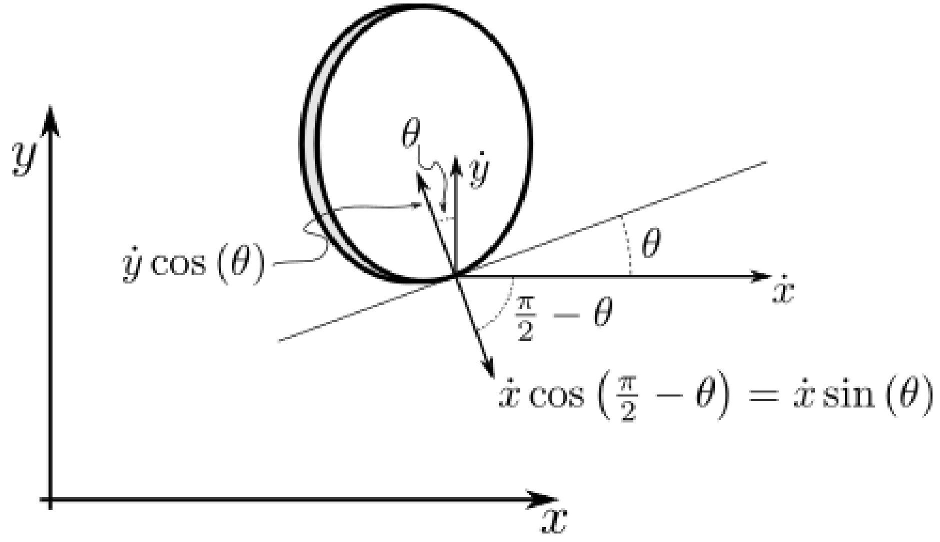
or in Pfaffian form

$$a^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix}\sin(\theta) & -\cos(\theta) & 0\end{bmatrix}\dot{\mathbf{q}} = 0$$

Determine the nonholonomic constraint characterising the motion of a rolling disk

Despite the constraint, the rolling disk can reach any position (x, y) in the motion plane, with any orientation θ.

The pure rolling constraint is thus a nonholonomic constraint.



How can we decide if a general kinematic constraint is holonomic / nonholonomic without making reference to intuition?

See additional material A for the derivation

Let's consider a 3D configuration space $\boldsymbol{q} = [x \quad y \quad z]^T$.

A Pfaffian constraint

$$a^T(\mathbf{q})\dot{\mathbf{q}} = [X(\mathbf{q}) \quad Y(\mathbf{q}) \quad Z(\mathbf{q})]\dot{\mathbf{q}} = X(\mathbf{q})\dot{x} + Y(\mathbf{q})\dot{y} + Z(\mathbf{q})\dot{z} = 0$$

is holonomic if it can be reduced to the form

$$h(\mathbf{q}) = h(x, y, z) = 0$$

For this constraint to be integrable, it is *necessary and sufficient* that there exist an integrating factor $\alpha(\boldsymbol{q}) = \alpha(x, y, z)$, such that

$$\alpha(\mathbf{q})X(\mathbf{q})\dot{x} + \alpha(\mathbf{q})Y(\mathbf{q})\dot{y} + \alpha(\mathbf{q})Z(\mathbf{q})\dot{z} = 0$$

be an exact differential…

…and if it is an exact differential, there must exist a function Γ, such that

$$\alpha(\mathbf{q})X(\mathbf{q}) = \frac{\partial \Gamma}{\partial x} \qquad \alpha(\mathbf{q})Y(\mathbf{q}) = \frac{\partial \Gamma}{\partial y} \qquad \alpha(\mathbf{q})Z(\mathbf{q}) = \frac{\partial \Gamma}{\partial z}$$

Pfaffian constraints: integrability conditions

Finally, the *necessary and sufficient* condition for the existence of function Γ, is that the first partial derivatives of $X(\boldsymbol{q})$, $Y(\boldsymbol{q})$, $Z(\boldsymbol{q})$, with respect to x, y, and z exist, and

$$\frac{\partial\left(\alpha\left(\mathbf{q}\right)X\left(\mathbf{q}\right)\right)}{\partial y} = \frac{\partial\left(\alpha\left(\mathbf{q}\right)Y\left(\mathbf{q}\right)\right)}{\partial x}$$

$$\frac{\partial\left(\alpha\left(\mathbf{q}\right)X\left(\mathbf{q}\right)\right)}{\partial z} = \frac{\partial\left(\alpha\left(\mathbf{q}\right)Z\left(\mathbf{q}\right)\right)}{\partial x}$$

$$\frac{\partial\left(\alpha\left(\mathbf{q}\right)Z\left(\mathbf{q}\right)\right)}{\partial y} = \frac{\partial\left(\alpha\left(\mathbf{q}\right)Y\left(\mathbf{q}\right)\right)}{\partial z}$$

Is this constraint holonomic or nonholonomic?

Consider the following kinematic constraint

$$\dot{x} + x\dot{y} + \dot{z} = 0$$

It can be rewritten in the following form

$$a^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = 0$$

where $X(\boldsymbol{q}) = 1$, $Y(\boldsymbol{q}) = x$, $Z(\boldsymbol{q}) = 1$.

Applying the previous relations we obtain…

Is this constraint holonomic or nonholonomic?

Applying the previous relations we obtain

$$X(\boldsymbol{q}) = 1 \qquad Y(\boldsymbol{q}) = x \qquad Z(\boldsymbol{q}) = 1$$

$$\frac{\partial(\alpha(\mathbf{q})X(\mathbf{q}))}{\partial y} = \frac{\partial(\alpha(\mathbf{q})Y(\mathbf{q}))}{\partial x} \qquad\qquad \frac{\partial \alpha(\mathbf{q})}{\partial y} = \frac{\partial(\alpha(\mathbf{q})x)}{\partial x} = \alpha(\mathbf{q}) + x\frac{\partial \alpha(\mathbf{q})}{\partial x}$$

$$\frac{\partial(\alpha(\mathbf{q})X(\mathbf{q}))}{\partial z} = \frac{\partial(\alpha(\mathbf{q})Z(\mathbf{q}))}{\partial x} \quad \Rightarrow \quad \frac{\partial \alpha(\mathbf{q})}{\partial z} = \frac{\partial \alpha(\mathbf{q})}{\partial x}$$

$$\frac{\partial(\alpha(\mathbf{q})Z(\mathbf{q}))}{\partial y} = \frac{\partial(\alpha(\mathbf{q})Y(\mathbf{q}))}{\partial z} \qquad\qquad \frac{\partial \alpha(\mathbf{q})}{\partial y} = \frac{\partial(\alpha(\mathbf{q})x)}{\partial z} = x\frac{\partial \alpha(\mathbf{q})}{\partial z}$$

Substituting now the second and third relation into the first one we obtain

$$x\frac{\partial \alpha(\mathbf{q})}{\partial z} = \alpha(\mathbf{q}) + x\frac{\partial \alpha(\mathbf{q})}{\partial z} \qquad \Rightarrow \qquad \alpha(\mathbf{q}) = 0$$

We thus conclude that the constraint is nonholonomic.

Integrability conditions: extension to n dimensions

The previous results can be easily extended to an n-dimensional configuration space, i.e., $\boldsymbol{q} \in \mathbb{R}^n$.

Consider a single Pfaffian constraint

$$a^T(\mathbf{q})\,\dot{\mathbf{q}} = \sum_{j=1}^{n} a_j(\mathbf{q})\,\dot{q}_j = 0$$

The necessary and sufficient integrability conditions may be replaced by the following system of partial differential equations

$$\frac{\partial(\alpha(\mathbf{q})\,a_k(\mathbf{q}))}{\partial q_j} = \frac{\partial(\alpha(\mathbf{q})\,a_j(\mathbf{q}))}{\partial q_k} \qquad j,k = 1,\ldots,n \quad j \neq k$$

Integrability conditions: extension to k constraints

If we consider a set of $k > 1$ kinematic constraints the definition of integrability conditions is far more complex.

The single constraints may be not integrable if taken separately, but the whole set can be integrable.

Consider the following system of Pfaffian constraints

$$\dot{q}_1 + q_1\dot{q}_2 + \dot{q}_3 = 0$$

$$\dot{q}_1 + \dot{q}_2 + q_1\dot{q}_3 = 0$$

we already know that the first constraint is nonholonomic, what about the all system?

As the constraints must be satisfied for any value of $\boldsymbol{q}$, we can substitute the first constraint with the difference between the two

$$(q_1 - 1)(\dot{q}_2 - \dot{q}_3) = 0 \qquad \Rightarrow \qquad \dot{q}_2 = \dot{q}_3$$

The new system of constraints is thus

$$\dot{q}_2 = \dot{q}_3$$
$$\dot{q}_1 + (1 + q_1)\,\dot{q}_2 = 0$$

The two constraints can be integrated, obtaining

$$q_2 - q_3 = c_1$$
$$\log(1 + q_1) + q_2 = c_2$$

with integration constants c_1 and c_2.

Using kinematic constraints to derive a kinematic model

We can interpret a system of k kinematic constraints

$$A^T(\mathbf{q})\dot{\mathbf{q}} = 0$$

as follows:

for any configuration $\boldsymbol{q}$ the admissible generalized velocities $\dot{\boldsymbol{q}}$ belong to the $(n-k)$-dimensional null space of matrix $A^T(\boldsymbol{q})$.

Denoting by

$$\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q}) \ldots, \mathbf{g}_{n-k}(\mathbf{q})\}$$

a basis of $\mathrm{Null}(A^T(\boldsymbol{q}))$, the admissible trajectories of the mechanical system are solutions of the nonlinear dynamic system

$$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q}) u_j = G(\mathbf{q})\mathbf{u} \qquad m = n-k$$

We thus call the dynamic system

$$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q}) u_j = G(\mathbf{q})\mathbf{u} \qquad m = n - k$$

kinematic model of the mechanical system.

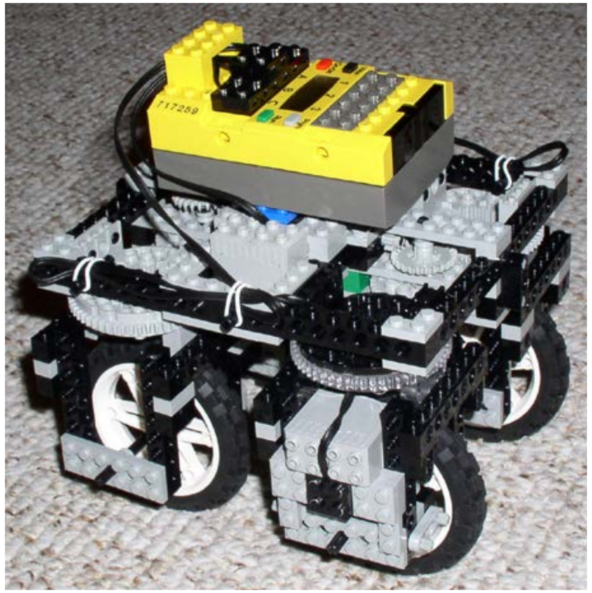
A few remarks:

- the choice of vectors $\boldsymbol{g}_1(\boldsymbol{q})$, …, $\boldsymbol{g}_m(\boldsymbol{q})$ is not unique
- the basis of $\mathrm{Null}\left(A^T(\boldsymbol{q})\right)$ can be selected in a way that input u_j have a physical interpretation
- holonomy / nonholonomy of constraints can be assessed analyzing the controllability properties of the kinematic model (a nonlinear dynamic model!)

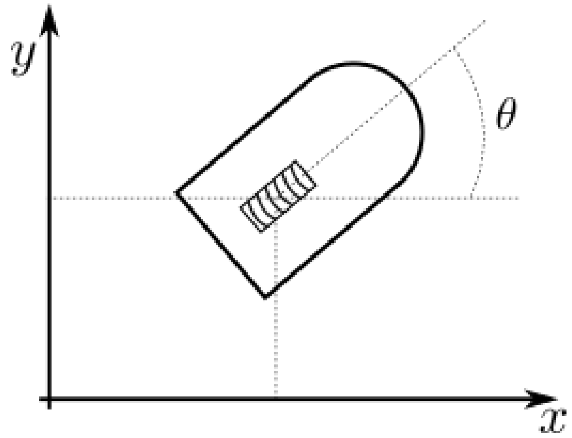
Deriving the kinematic model of a unicycle vehicle

We would like to derive the constraints that characterize the motion of these vehicles, and their kinematic models.

All these vehicles are kinematically equivalent to a unicycle vehicle.
Let’s derive the constraints and the kinematic model of a unicycle.



Deriving the kinematic model of a unicycle vehicle



A unicycle is a vehicle with a single orientable wheel.

Its configuration is described by the position of the wheel contact point and the wheel orientation

$$\mathbf{q} = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$$

We have already introduced the pure rolling constraint describing a wheel

$$a^T(\mathbf{q})\dot{\mathbf{q}} = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 \end{bmatrix}\dot{\mathbf{q}} = 0$$

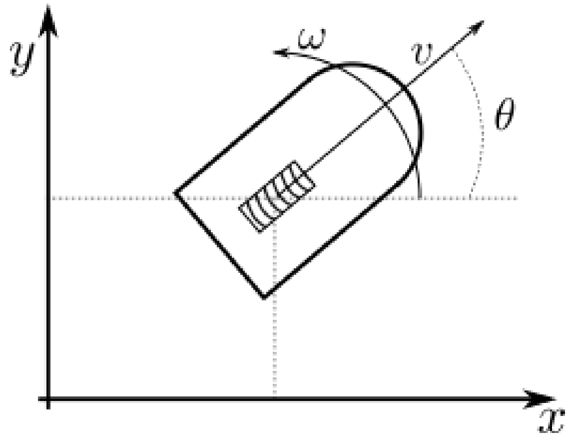
A basis of the null space of matrix $a^T(\boldsymbol{q})$ is

$$\left\{ \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \Rightarrow \quad G(\mathbf{q}) = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

Deriving the kinematic model of a unicycle vehicle

The kinematic model of the unicycle can be thus expressed as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

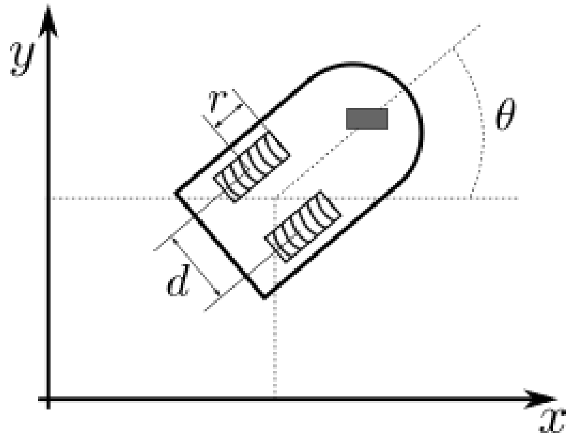


u_1 and u_2 have a straightforward physical interpretation, they are the *linear and angular velocity* of the vehicle.

The kinematic model can be thus expressed as

$$\begin{cases} \dot{x} = v\cos(\theta) \\ \dot{y} = v\sin(\theta) \\ \dot{\theta} = \omega \end{cases}$$

Deriving the kinematic model of a differential drive vehicle



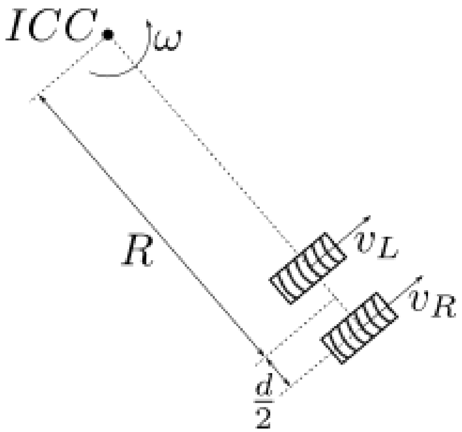
In a differential drive vehicle two wheels are actuated by independent motors, having a rotational velocity ω_R and ω_L.

The linear and angular velocity of the vehicle are thus related to wheel velocities by simple kinematic considerations.

First, each wheel has a linear velocity $v_R = \omega_R r$ and $v_L = \omega_L r$, the average value between the two velocities represents the vehicle linear velocity

$$v = r\frac{\omega_R + \omega_L}{2}$$

A difference in the velocity of the two wheels, instead, generates a rotation of the vehicle around a point lying on the wheel axis, called *Instantaneous Center of Curvature* (ICC).



Because the rate of rotation ω around the *ICC* must be the same for both wheels, we can write the following equations

$$\omega\left(R+\frac{d}{2}\right)=v_R$$

$$\omega\left(R-\frac{d}{2}\right)=v_L$$

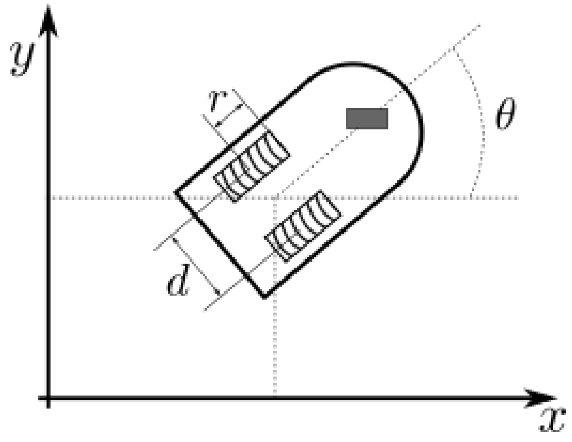
Solving now with respect to ω and R, we obtain

$$\omega=\frac{v_R-v_L}{d}=r\frac{\omega_R-\omega_L}{d} \qquad R=\frac{d}{2}\frac{v_R+v_L}{v_R-v_L}$$

Summarizing...

...the kinematic model of a differential drive vehicle is

$$\begin{cases} \dot{x} = \frac{\omega_R + \omega_L}{2} r \cos(\theta) \\ \dot{y} = \frac{\omega_R + \omega_L}{2} r \sin(\theta) \\ \dot{\theta} = \frac{\omega_R - \omega_L}{d} r \end{cases}$$

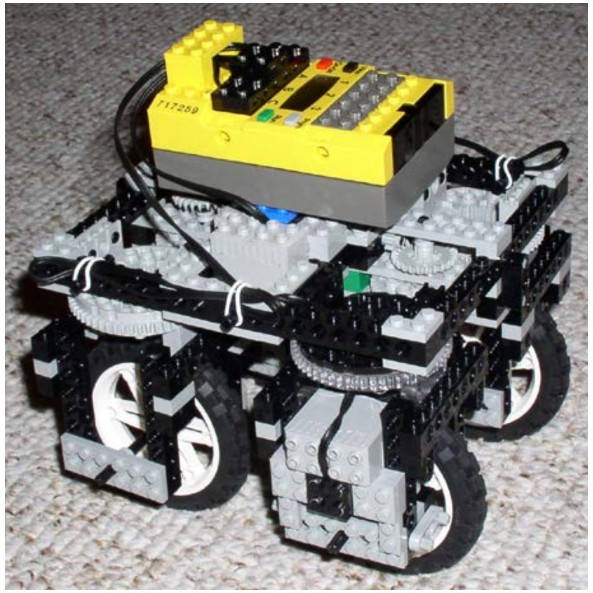


Deriving the kinematic model of a synchro drive vehicle

In the synchro drive robot all the wheels rotate at the same velocity, the correspondence with the unicycle model is thus straightforward.

x and y represent any point of the robot, for example the centroid, θ is the common orientation of the wheels.

In this case to change the orientation of the robot a further actuator must be added.



Deriving the kinematic model of a bicycle vehicle

We consider now another family of vehicles, for which we would like to derive the constraints that characterize the motion and their kinematic models.

All these vehicles are kinematically equivalent to a bicycle vehicle. Let's derive the constraints and the kinematic model of a bicycle.

A bicycle is a vehicle with an orientable wheel and a fixed wheel.

Its configuration is described by the position of the rear wheel contact point, the orientation of the vehicle and the steering angle

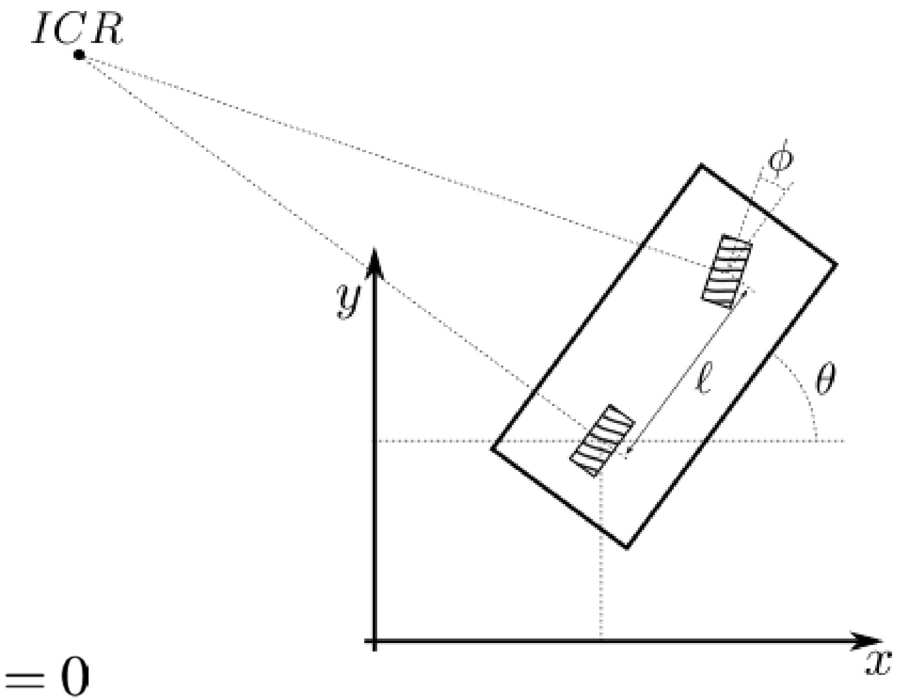
$$\mathbf{q} = \begin{bmatrix} x & y & \theta & \phi \end{bmatrix}^T$$

The mechanical system is characterized by two pure rolling constraints, one for each wheel

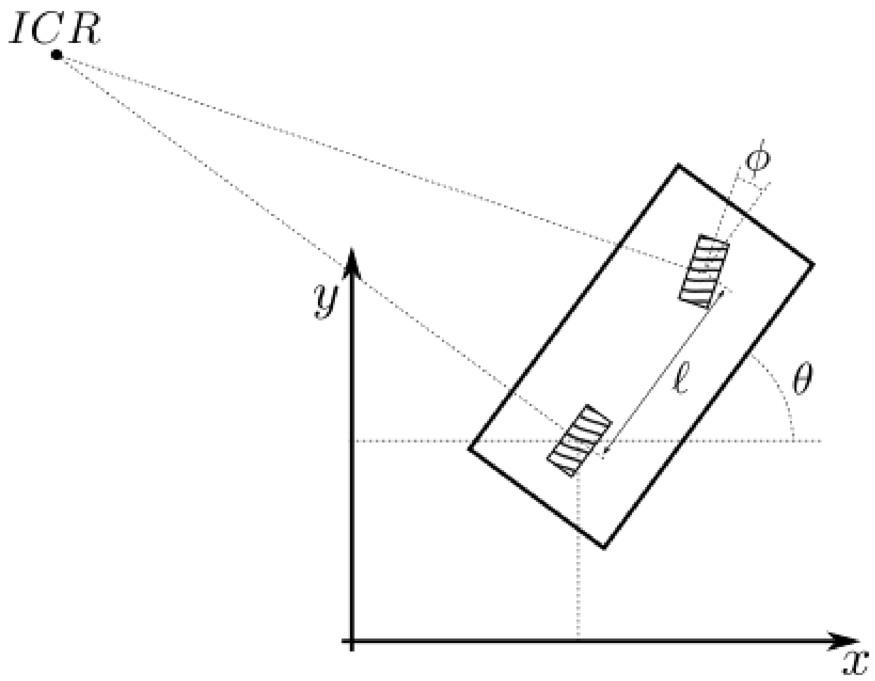
$$\dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) = 0$$

$$\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = 0$$

where (x_f, y_f) is the position of the center of the front wheel.



Each constraint defines a *zero motion line*.
The two zero motion lines meet at a point called *Instantaneous Center of Rotation* (ICR), whose position depends only on the configuration of the vehicle.
Each point of the bicycle rotates instantaneously around the ICR.



The positions of the wheels are related by

$$x_f = x + \ell \cos(\theta)$$

$$y_f = y + \ell \sin(\theta)$$

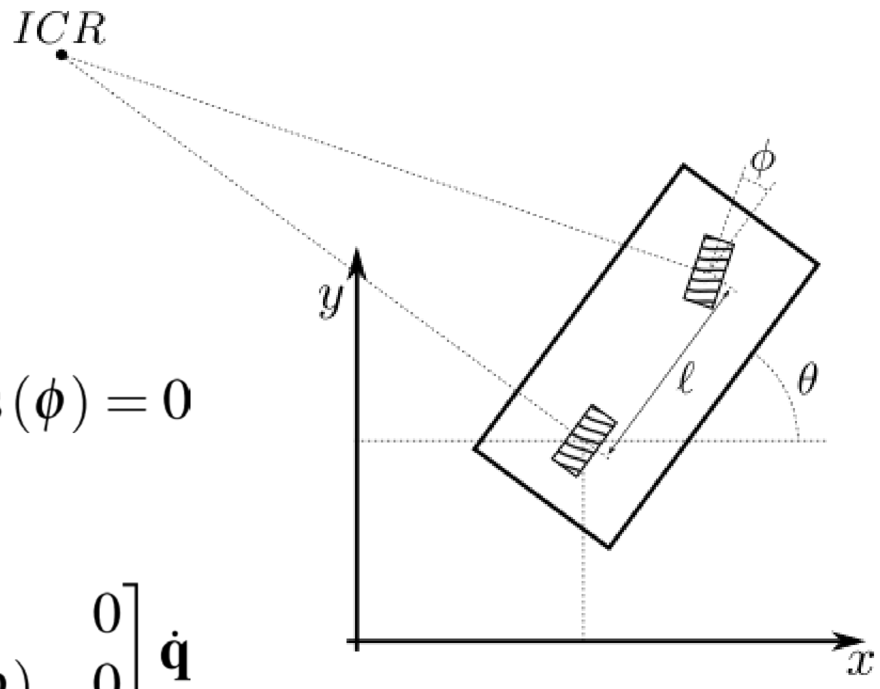
The front wheel constraint can be thus expressed as

$$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - \ell\dot{\theta}\cos(\phi) = 0$$

The two constraints can be now expressed in Pfaffian form

$$A^T(\mathbf{q})\dot{\mathbf{q}} = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos(\phi) & 0 \end{bmatrix}\dot{\mathbf{q}}$$

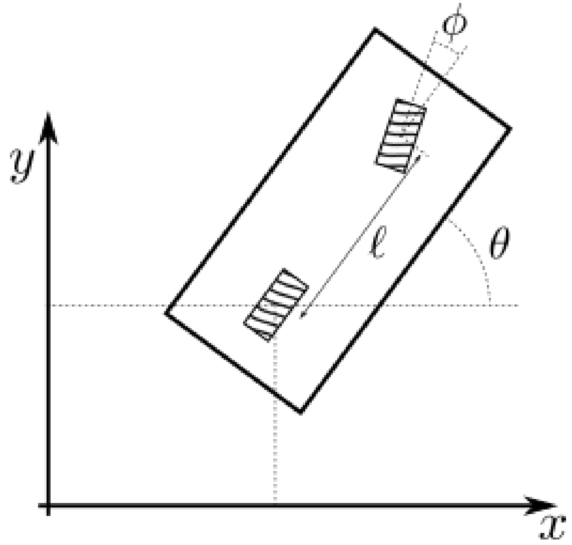
We can observe that matrix $A(\boldsymbol{q})$ has always rank equal to 2, and a null space of dimension 2.



Deriving the kinematic model of a bicycle vehicle

The admissible generalized velocities can be written as a linear combination of the basis of $\mathrm{Null}\left(A^T(\boldsymbol{q})\right)$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ \sin(\phi)/\ell \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$



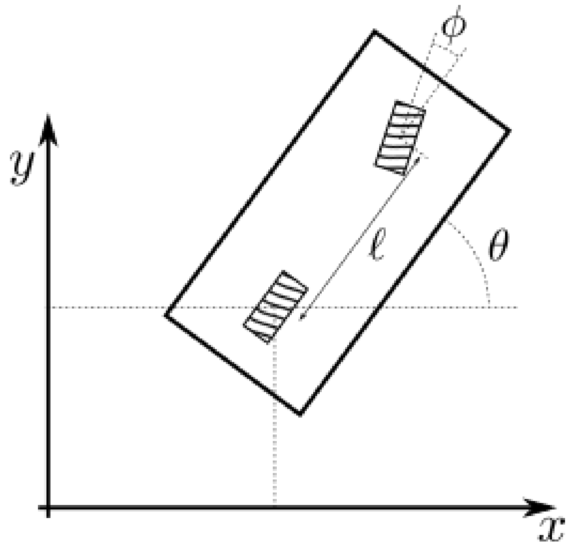
What is the physical interpretation of the inputs?

- $u_2 = \omega$ is the rate of change of the steering wheel
- u_1 depends on how the vehicle is driven

Deriving the kinematic model of a bicycle vehicle

For *front-wheel drive* bicycles, $u_1 = v$ is the driving velocity of the front wheel

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ \sin(\phi)/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$



For *rear-wheel drive* bicycles, the first two equations must coincide with those of the unicycle model. We must select $u_1 = v/\cos\phi$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \tan(\phi)/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

Deriving the kinematic model of a quadrotor and of an underwater spherical vehicle

Going back to the reasons that motivate the introduction of kinematic constraints as a tool to derive the kinematic model of a robot, we would like to apply this tool to

- a quadrotor
- an underwater spherical vehicle

The motion of these robots is not based on wheels, and they belong to application domains (aerial and underwater robotics) that are completely different from ground robotics.

Kinematic constraints are thus a powerful and general tool to derive kinematic models of mobile robots.

Deriving the kinematic model of a quadrotor

A quadrotor is a mobile robot actuated by four (or more) propellers.

The quadrotor configuration is described by the pose of its body frame with respect to a world reference frame

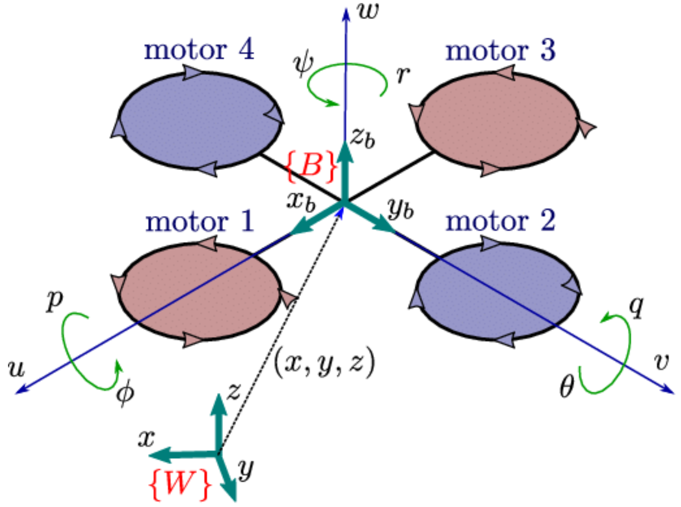
$$\mathbf{q} = \begin{bmatrix} x & y & z & \varphi & \theta & \psi \end{bmatrix}^T$$

To simplify the derivation we assume a constant yaw angle equal to zero, the configuration vector reduces to

$$\mathbf{q} = \begin{bmatrix} x & y & z & \varphi & \theta \end{bmatrix}^T$$

Considering that propellers are able to generate only

- force in a direction parallel the their rotation axis
- moments around the body frame axis

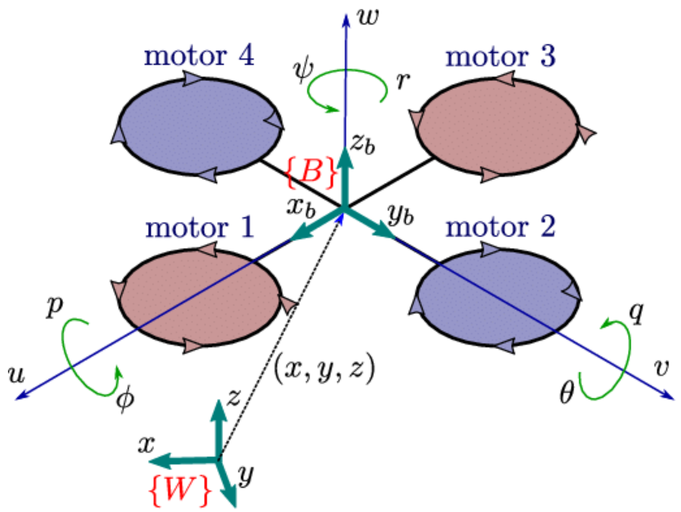


Considering that propellers are able to generate only

- force in a direction parallel the their rotation axis
- moments around the body frame axis

we can introduce the following constraints

$$\dot{x}_b = 0 \qquad \dot{y}_b = 0$$



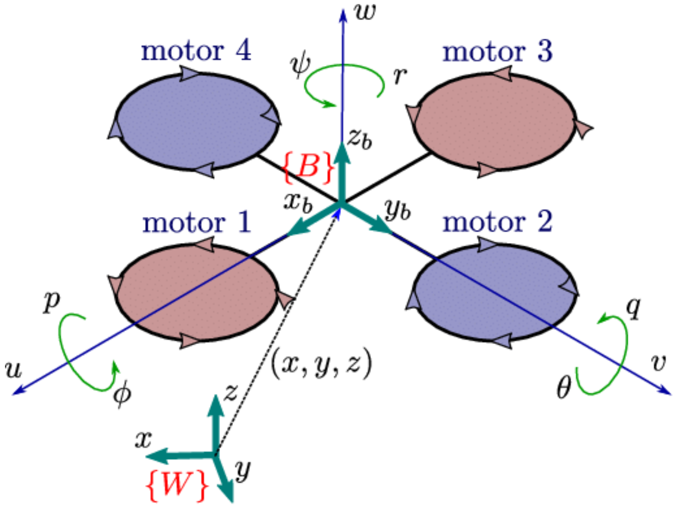
In order to express these constraints in the world reference frame, as a function of the configuration vector, we have to introduce the rotation matrix relating the body to the word reference frame

$${}^{w}R_b = \begin{bmatrix} \cos\psi\cos\theta - \sin\varphi\sin\psi\sin\theta & -\cos\varphi\sin\psi & \cos\psi\sin\theta + \sin\varphi\sin\psi\cos\theta \\ \sin\varphi\cos\psi\sin\theta + \sin\psi\cos\theta & \cos\varphi\cos\psi & \sin\psi\sin\theta - \sin\varphi\cos\psi\cos\theta \\ -\cos\varphi\sin\theta & \sin\varphi & \cos\varphi\cos\theta \end{bmatrix}$$

Deriving the kinematic model of a quadrotor

We can now write

$$\begin{bmatrix} \dot{x}_b \\ \dot{y}_b \\ \dot{z}_b \end{bmatrix} = {}^{w}R_b^T \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$



and the two constraints become

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} \cos\theta & \sin\varphi\sin\theta & -\cos\varphi\sin\theta & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 & 0 \end{bmatrix} \dot{\mathbf{q}} = 0$$

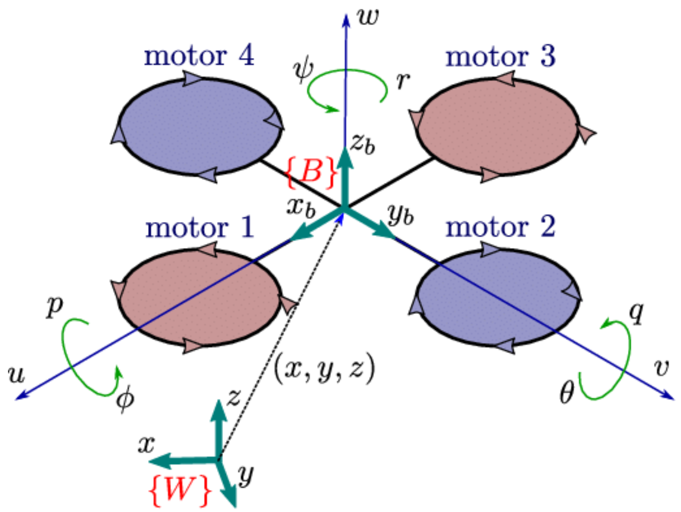
Matrix $A(\boldsymbol{q})$ has rank equal to 2 and a null space of dimension 3, a basis of the null space is

$$\left\{ \begin{bmatrix} \sin\theta \\ -\sin\varphi\cos\theta \\ \cos\varphi\cos\theta \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Deriving the kinematic model of a quadrotor

The kinematic model of the quadrotor can be thus expressed as

$$\dot{x} = v \sin\theta$$
$$\dot{y} = -v \sin\varphi \cos\theta$$
$$\dot{z} = v \cos\varphi \cos\theta$$
$$\dot{\varphi} = \omega_\varphi$$
$$\dot{\theta} = \omega_\theta$$



where v is the velocity in the direction parallel to the propeller axis.

Deriving the kinematic model of an underwater spherical vehicle

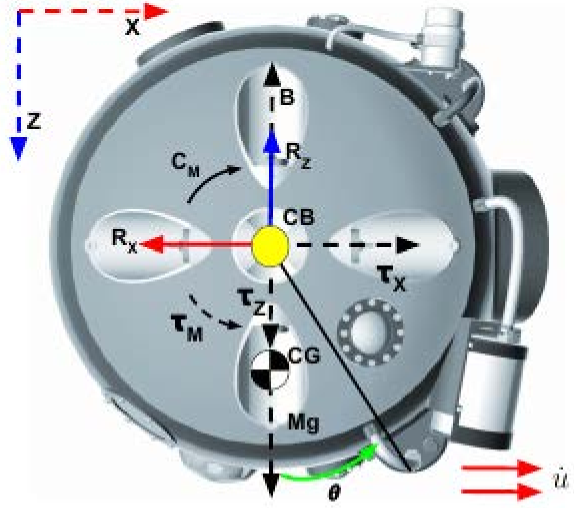
This underwater spherical vehicle is actuated by eight propellers that can generate a force in the forward direction, or rotate the robot around the axis of its body reference frame.

As for the quadrotor, vehicle configuration is described by the pose of its body frame with respect to a world reference frame

$$\mathbf{q} = \begin{bmatrix} x & y & z & \varphi & \theta & \psi \end{bmatrix}^T$$

In this case, to simplify the derivation we assume a constant roll angle equal to zero. The configuration vector reduces to

$$\mathbf{q} = \begin{bmatrix} x & y & z & \theta & \psi \end{bmatrix}^T$$



Considering that propellers are able to generate only

- force in the forward direction
- moments around the body frame axis

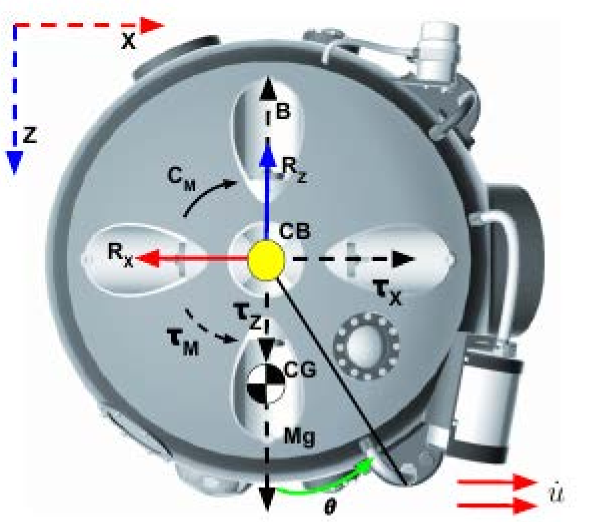
we can introduce the following constraints

$$\dot{y}_b = 0 \qquad \dot{z}_b = 0$$

Following the same procedure adopted for the quadrotor, we can rewrite these constraints with respect to the world reference frame, obtaining

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} -\sin\psi & \cos\psi & 0 & 0 & 0 \\ \cos\psi\sin\theta & \sin\psi\sin\theta & \cos\theta & 0 & 0 \end{bmatrix} \dot{\mathbf{q}} = 0$$

Matrix $A(\boldsymbol{q})$ has rank equal to 2 and a null space of dimension 3.



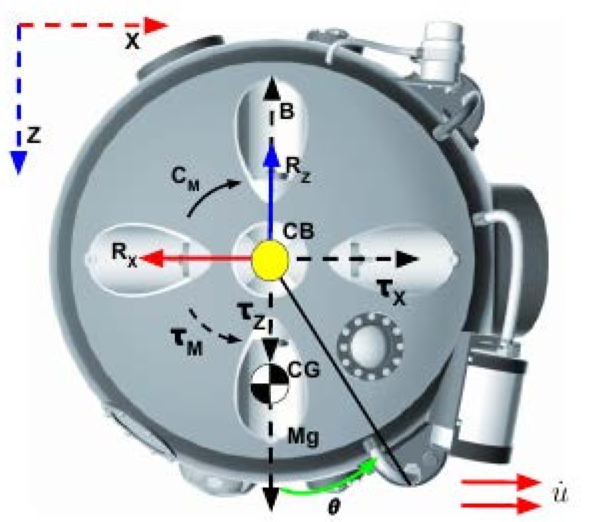
Deriving the kinematic model of an underwater spherical vehicle

A basis of the null space of matrix $A^T(\boldsymbol{q})$ is

$$\left\{ \begin{bmatrix} -\cos\psi\cos\theta \\ -\sin\psi\cos\theta \\ \sin\theta \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The kinematic model of the underwater vehicle can be expressed as

$$\begin{aligned} \dot{x} &= -v\cos\psi\cos\theta \\ \dot{y} &= -v\sin\psi\cos\theta \\ \dot{z} &= v\sin\theta \\ \dot{\psi} &= \omega_\psi \\ \dot{\theta} &= \omega_\theta \end{aligned}$$



where v is the velocity in the forward direction.

Let’s go back to the definition of holonomy / nonholonomy.

Holonomic constraints reduce the available DOF of the system, and thus the number of generalized coordinates used to describe its motion.

Holonomic constraints cause a loss of accessibility: there are points in the configuration space that are not reachable anymore.

Nonholonomic constraints locally limit the generalized velocities to belong to $Null\left(A^T(\boldsymbol{q})\right)$, but there is no constraint, neither locally, that allows to decrease the number of generalized coordinates.

Nonholonomic constraints do not cause any loss of accessibility: all the points in the configuration space are still reachable.

Do holonomy / nonholonomy resemble any well-known property of a dynamic system?

The dynamic system (kinematic model)

$$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q}) u_j = G(\mathbf{q})\mathbf{u} \qquad m = n - k$$

- is controllable if given two arbitrary configurations $\boldsymbol{q}_i$ and $\boldsymbol{q}_f$, there exists a choice of $\boldsymbol{u}$ that steers the system from $\boldsymbol{q}_i$ to $\boldsymbol{q}_f$;
- otherwise it is not controllable.

If the system is contrallable there exists a trajectory that joins any two arbitrary configurations and satisfies the kinematic constraints, all the configurations are thus reachable (there is no loss of accessibility), constraints are nonholonomic.

If the system is not controllable, the kinematic constraints reduce the set of accessible configurations, the constraints are thus partially or completely integrable (constraints can be holonomic or nonholonomic).

If the system is not controllable, depending on the dimension $\nu < n$ of the accessible configuration space it can be

- if $n - k < \nu < n$, the loss of accessibility is not maximal, constraints are only partially integrable. The mechanical system is still nonholonomic;
- if $\nu = n - k$, the loss of accessibility is maximal, and constraints are completely integrable. The mechanical system is holonomic.

We conclude that a more simple way to answer the question «is a set of k kinematic constraints holonomic or nonholonomic?» is by checking the controllability of the corresponding kinematic model.

Let's focus now on the relation between holonomy / nonholonomy and configuration accessibility...

We concentrate on the system that represents a kinematic model of a mobile robot

$$\dot{\mathbf{q}} = G(\mathbf{q})\,\mathbf{u} = \sum_{i=1}^{m} g_i(\mathbf{q})\,u_i$$

and we start considering small motions generated by each vector field g_i.

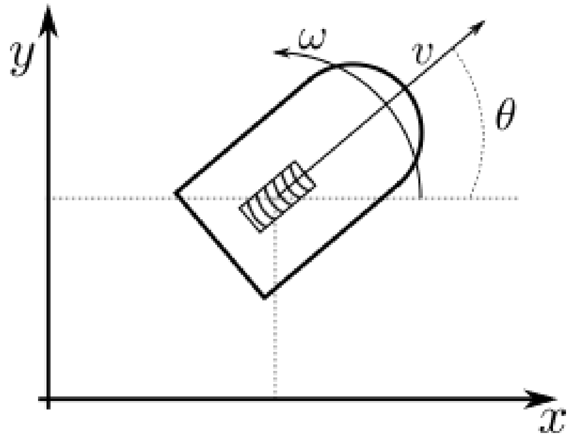
Do two vector fields commute?

$$F_{\varepsilon}^{g_j}\left(F_{\varepsilon}^{g_i}(\mathbf{q})\right) = F_{\varepsilon}^{g_i}\left(F_{\varepsilon}^{g_j}(\mathbf{q})\right) \quad ?$$

Let’s start from an example, considering the unicycle model…

Let's consider the unicycle model

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\cos(\theta) & 0\\ \sin(\theta) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix}$$



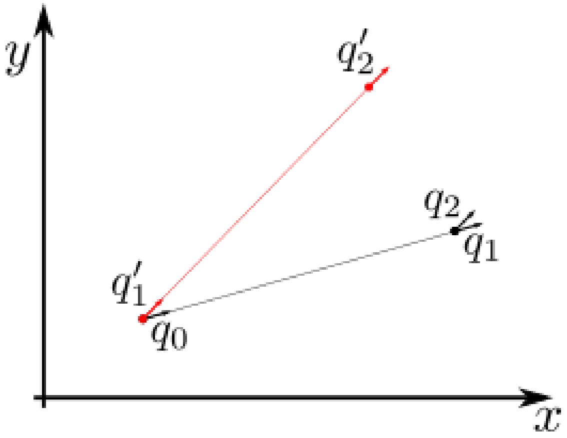
the two vector fields are

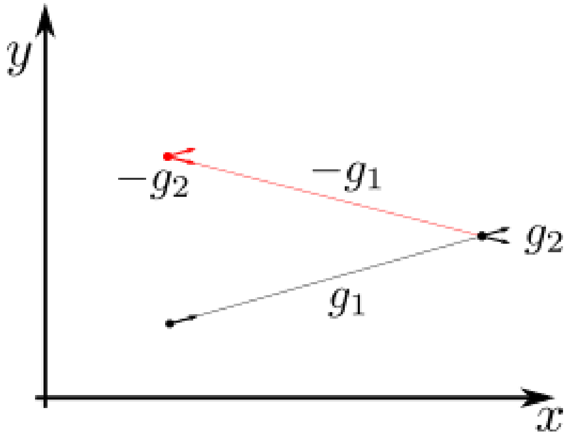
$$g_1(\mathbf{q}) = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\\ 0\end{bmatrix} \qquad g_2(\mathbf{q}) = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$$

g_1 represents a forward-backward motion in the direction defined by θ

g_2 represents a spin-in-place motion

Consider now an example of application of g_1 and g_2, and then g_2 and g_1, do we end at the same pose?

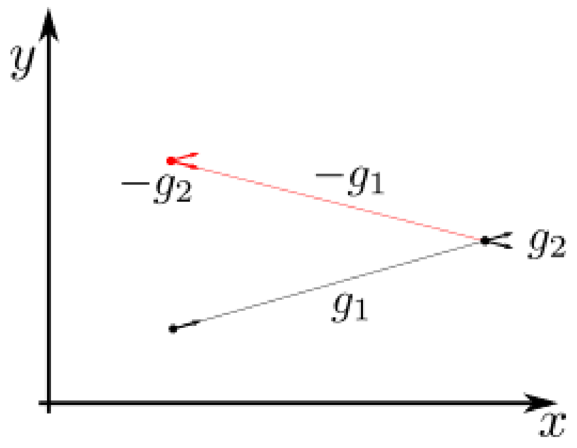




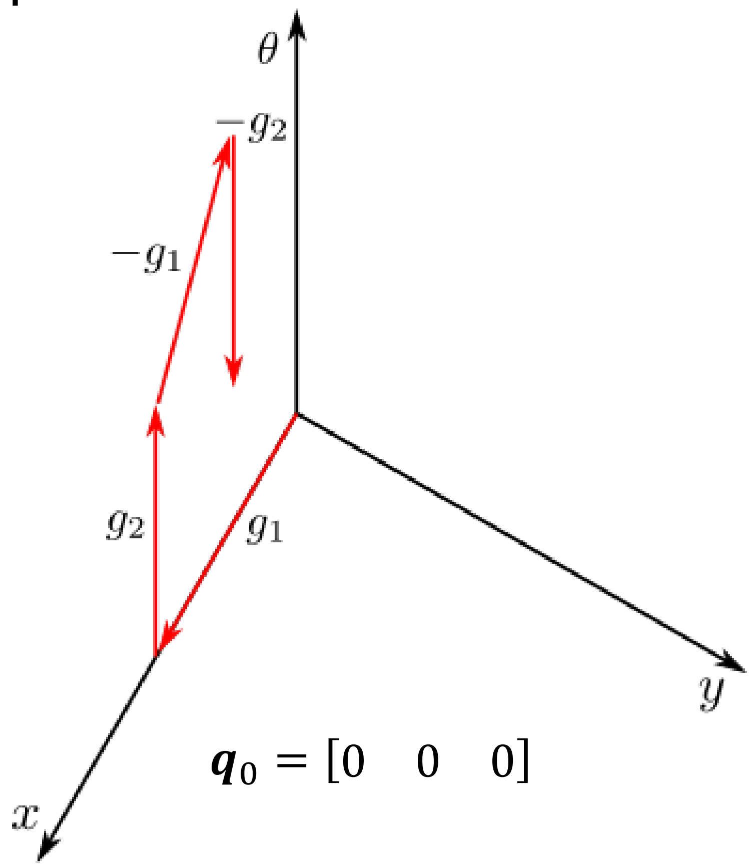
From these two examples we conclude that the two vector fields do not commute

$$F_{\varepsilon}^{g_j}\left(F_{\varepsilon}^{g_i}(\mathbf{q})\right) \neq F_{\varepsilon}^{g_i}\left(F_{\varepsilon}^{g_j}(\mathbf{q})\right)$$

We can see the previous example in the configuration space instead of the 2D robot workspace



$\boldsymbol{q}_0 = [x_0 \quad y_0 \quad \theta_0]$



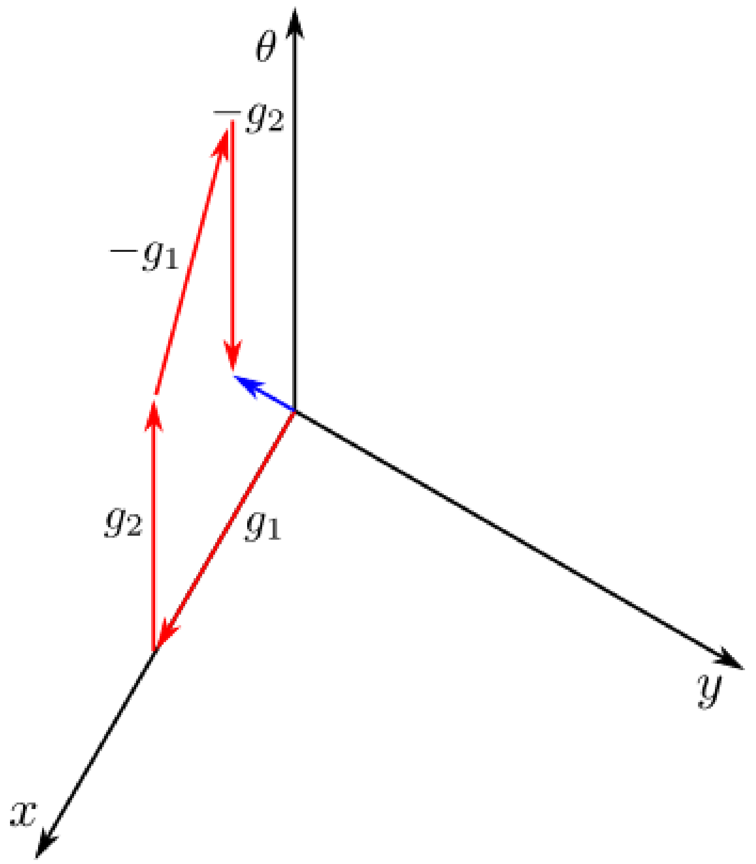
$\boldsymbol{q}_0 = [0 \quad 0 \quad 0]$

As the two vector fields do not commute, the result of the application of g_1 and g_2, and then $-g_1$ and $-g_2$ is a motion in a direction not present in the original set of vector fields.

We can define the noncommutativity in a way that shows this motion

$$\Delta \mathbf{q} = F_{\varepsilon}^{g_j}\left(F_{\varepsilon}^{g_i}(\mathbf{q})\right) - F_{\varepsilon}^{g_i}\left(F_{\varepsilon}^{g_j}(\mathbf{q})\right)$$

Now we would like to compute, at least approximately, the value of $\Delta\boldsymbol{q}$.



We consider two motions, the first along g_i and the second along g_j for a small time ϵ.

We start from $\boldsymbol{q}(0)$ and move along g_i for a small time ϵ. We approximate $\boldsymbol{q}(\epsilon)$ using a Taylor expansion truncated at $O(\epsilon^3)$

$$\mathbf{q}(\varepsilon) = \mathbf{q}(0) + \varepsilon\dot{\mathbf{q}}(0) + \frac{1}{2}\varepsilon^2\ddot{\mathbf{q}}(0) + O(\varepsilon^3)$$

Now we can observe that

$$\dot{\mathbf{q}} = g_i(\mathbf{q}) \qquad \ddot{\mathbf{q}} = \frac{\partial g_i}{\partial \mathbf{q}}\dot{\mathbf{q}} = \frac{\partial g_i}{\partial \mathbf{q}} g_i(\mathbf{q})$$

and thus

$$\mathbf{q}(\varepsilon) = \mathbf{q}(0) + \varepsilon g_i(\mathbf{q}(0)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_i}{\partial \mathbf{q}} g_i(\mathbf{q}(0)) + O(\varepsilon^3)$$

Now we move from $\boldsymbol{q}(\epsilon)$ along g_j for a small time ϵ

$$\mathbf{q}(2\varepsilon) = \mathbf{q}(\varepsilon) + \varepsilon g_j(\mathbf{q}(\varepsilon)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_j}{\partial \mathbf{q}} g_j(\mathbf{q}(\varepsilon)) + O(\varepsilon^3)$$

but we know that

$$\mathbf{q}(\varepsilon) = \mathbf{q}(0) + \varepsilon g_i(\mathbf{q}(0)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_i}{\partial \mathbf{q}} g_i(\mathbf{q}(0)) + O(\varepsilon^3)$$

and thus

$$\begin{aligned}\mathbf{q}(2\varepsilon) = \quad & \mathbf{q}(0) + \varepsilon g_i(\mathbf{q}(0)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_i}{\partial \mathbf{q}} g_i(\mathbf{q}(0)) \\ & + \varepsilon g_j(\mathbf{q}(0) + \varepsilon g_i(\mathbf{q}(0))) + \frac{1}{2}\varepsilon^2 \frac{\partial g_j}{\partial \mathbf{q}} g_j(\mathbf{q}(0)) + O(\varepsilon^3)\end{aligned}$$

We can now simplify expression

$$\mathbf{q}(2\varepsilon) = \mathbf{q}(0) + \varepsilon g_i(\mathbf{q}(0)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_i}{\partial \mathbf{q}} g_i(\mathbf{q}(0)) + \boxed{\varepsilon g_j(\mathbf{q}(0) + \varepsilon g_i(\mathbf{q}(0)))} + \frac{1}{2}\varepsilon^2 \frac{\partial g_j}{\partial \mathbf{q}} g_j(\mathbf{q}(0)) + O(\varepsilon^3)$$

as follows

$$\mathbf{q}(2\varepsilon) = \mathbf{q}(0) + \varepsilon g_i(\mathbf{q}(0)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_i}{\partial \mathbf{q}} g_i(\mathbf{q}(0)) + \boxed{\varepsilon g_j(\mathbf{q}(0)) + \varepsilon^2 \frac{\partial g_j}{\partial \mathbf{q}} g_i(\mathbf{q}(0))} + \frac{1}{2}\varepsilon^2 \frac{\partial g_j}{\partial \mathbf{q}} g_j(\mathbf{q}(0)) + O(\varepsilon^3)$$

A system theory interpretation of holonomy and nonholonomy

Let's compare now $\boldsymbol{q}(2\epsilon)$ obtained following g_i and then g_j, or g_j and then g_i

$$\mathbf{q}_{g_i,g_j}(2\varepsilon) = \mathbf{q}(0) + \varepsilon g_i(\mathbf{q}(0)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_i}{\partial \mathbf{q}} g_i(\mathbf{q}(0)) + \varepsilon g_j(\mathbf{q}(0)) + \varepsilon^2 \frac{\partial g_j}{\partial \mathbf{q}} g_i(\mathbf{q}(0)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_j}{\partial \mathbf{q}} g_j(\mathbf{q}(0)) + O(\varepsilon^3)$$

$$\mathbf{q}_{g_j,g_i}(2\varepsilon) = \mathbf{q}(0) + \varepsilon g_j(\mathbf{q}(0)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_j}{\partial \mathbf{q}} g_j(\mathbf{q}(0)) + \varepsilon g_i(\mathbf{q}(0)) + \varepsilon^2 \frac{\partial g_i}{\partial \mathbf{q}} g_j(\mathbf{q}(0)) + \frac{1}{2}\varepsilon^2 \frac{\partial g_i}{\partial \mathbf{q}} g_i(\mathbf{q}(0)) + O(\varepsilon^3)$$

We conclude that

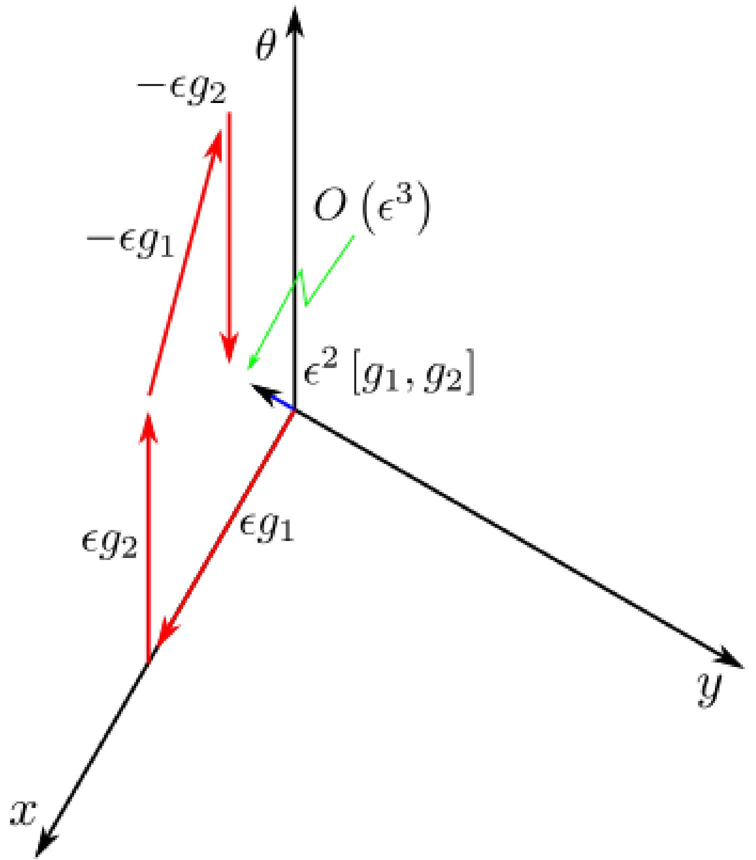
$$\Delta \mathbf{q} = \varepsilon^2 \left(\frac{\partial g_j}{\partial \mathbf{q}} g_i - \frac{\partial g_i}{\partial \mathbf{q}} g_j \right) (\mathbf{q}(0)) + O\left(\varepsilon^3\right)$$

This displacement represent the net motion obtained by following g_i for time ϵ, then g_j for time ϵ, then $-g_i$ for time ϵ, then $-g_j$ for time ϵ.

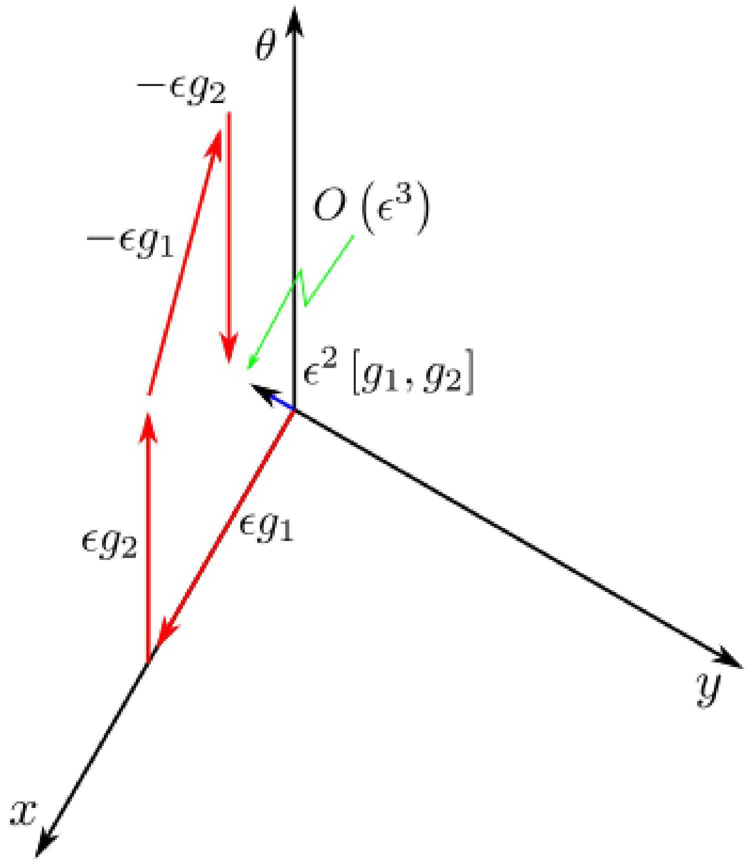
As a consequence, $\Delta \boldsymbol{q}$ is a vector field as well.

Given the two vector fields $g_i(\boldsymbol{q})$ and $g_j(\boldsymbol{q})$, we call the operation that allows to compute the new vector field *Lie bracket*

$$[g_i, g_j](\mathbf{q}) = \left(\frac{\partial g_j}{\partial \mathbf{q}} g_i - \frac{\partial g_i}{\partial \mathbf{q}} g_j \right) (\mathbf{q})$$



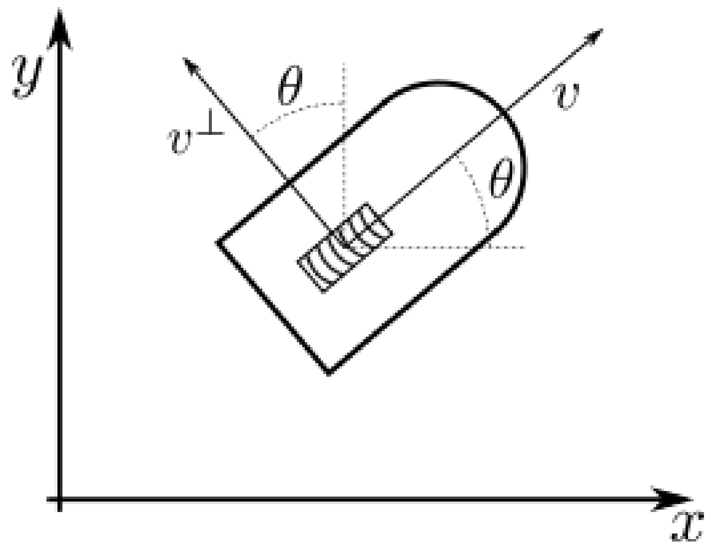
Motion along the Lie bracket vector field is slower (it is of order ϵ^2) with respect to the motions along the original vector fields (that are of order ϵ).



Let's go back again to the unicycle model.
The two vector fields are

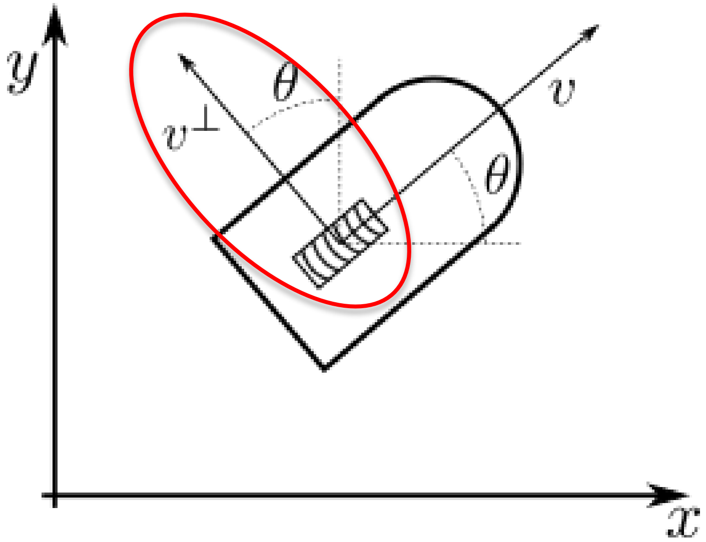
$$g_1(\mathbf{q}) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} \qquad g_2(\mathbf{q}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

What about the Lie bracket vector field?



Unicycle motion vector fields

$$g_1(\mathbf{q}) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} \qquad g_2(\mathbf{q}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



What about the Lie bracket vector field?

$$g_3(\mathbf{q}) = [g_1, g_2](\mathbf{q}) = \left(\frac{\partial g_2}{\partial \mathbf{q}} g_1 - \frac{\partial g_1}{\partial \mathbf{q}} g_2\right)(\mathbf{q})$$

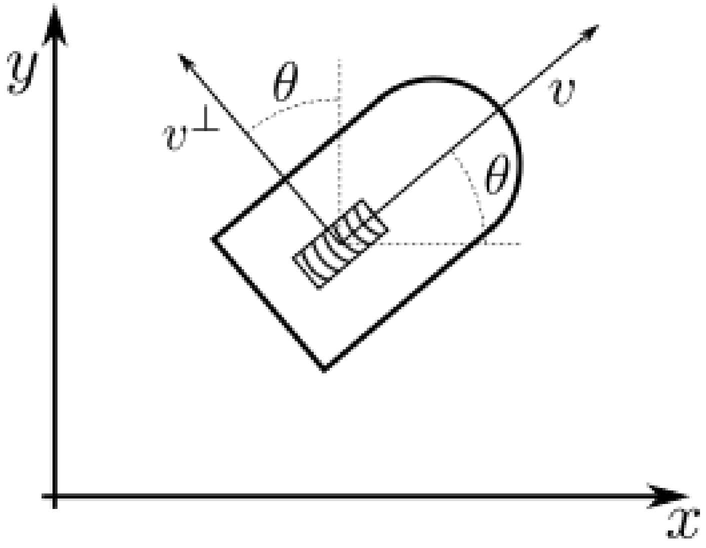
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -\sin(\theta) \\ 0 & 0 & \cos(\theta) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \end{bmatrix}$$

This represents a sideway motion

This result demonstrates that the unicycle motion is characterized by the composition of three different vector fields:

- g_1 and g_2 can be performed acting directly on v and ω
- g_3 can be performed through a sequence of manoeuvers generated acting on v and ω



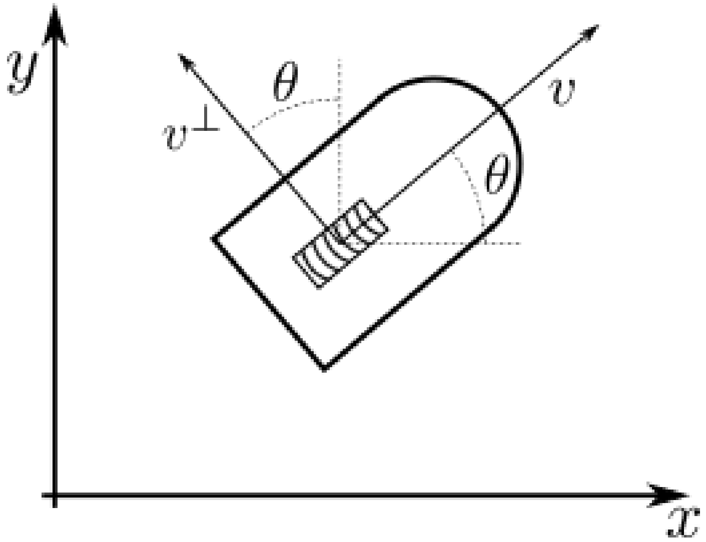
Are these three vector fields linearly independent?

$$\det\left(\begin{bmatrix} g_1(\mathbf{q}) & g_2(\mathbf{q}) & g_3(\mathbf{q}) \end{bmatrix}\right) = \det\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ \sin(\theta) & 0 & -\cos(\theta) \\ 0 & 1 & 0 \end{bmatrix} = \cos^2(\theta) + \sin^2(\theta) = 1$$

Yes, they are… a combination of these vectors allow to reach any configuration.

We can thus conclude that

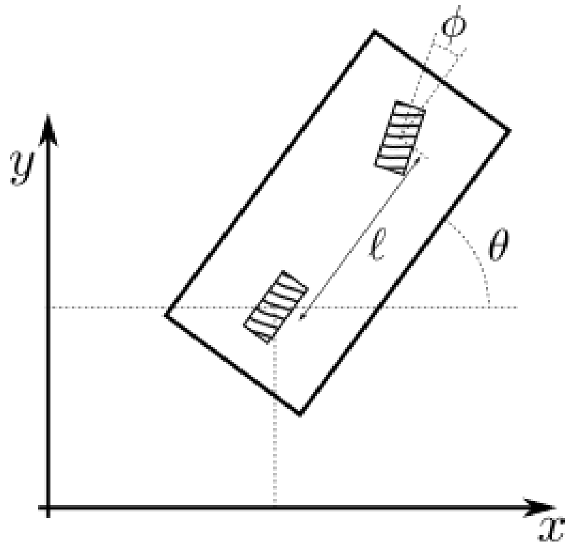
- there is no loss of accessibility
- $n = 3$ and $\dim \Delta_A = 3$
- the unicycle kinematic model is controllable
- the constraints are nonholonomic



Bicycle motion vector fields

Let's consider now the bicycle model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ \sin(\phi)/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$



the two vector fields are

$$g_1(\mathbf{q}) = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ \sin(\phi)/\ell \\ 0 \end{bmatrix} \qquad g_2(\mathbf{q}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

In this case, $n = 4$, to have controllability we need to find four independent vector fields, two more...

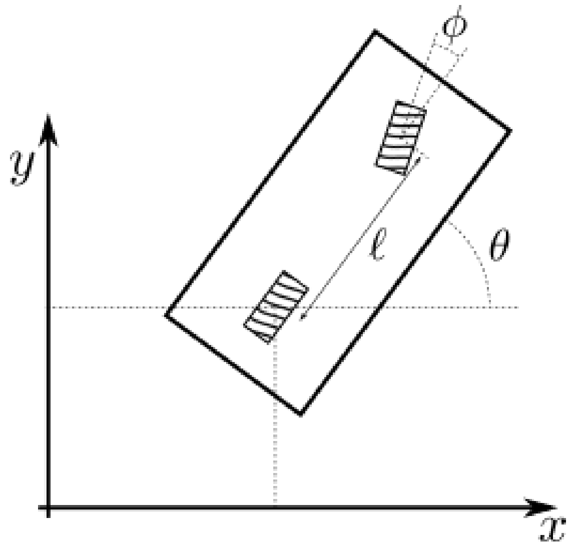
Bicycle motion vector fields

The Lie brackets allows to add a third vector field

$$g_3(\mathbf{q}) = [g_1, g_2](\mathbf{q}) = \begin{bmatrix} \cos(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) \\ -\cos(\phi)/\ell \\ 0 \end{bmatrix}$$

The last vector field can be generated using another Lie bracket

$$g_4(\mathbf{q}) = [g_1, g_3](\mathbf{q}) = \begin{bmatrix} -\sin(\theta)/\ell \\ \cos(\theta)/\ell \\ 0 \\ 0 \end{bmatrix}$$



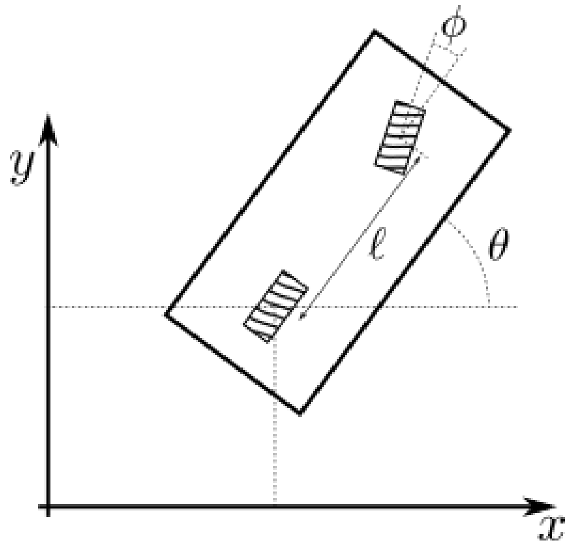
Are the four vector fields linearly independent?

Yes, the four vector fields are linearly independet

$$\det\left(\left[g_1(\mathbf{q})\ g_2(\mathbf{q})\ g_3(\mathbf{q})\ g_4(\mathbf{q})\right]\right) = \det\begin{bmatrix} \cos(\theta)\cos(\phi) & 0 & \cos(\theta)\sin(\phi) & -\sin(\theta)/\ell \\ \sin(\theta)\cos(\phi) & 0 & \sin(\theta)\sin(\phi) & \cos(\theta)/\ell \\ \sin(\phi)/\ell & 0 & -\cos(\theta)/\ell & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \frac{1}{\ell^2}$$

We can thus conclude that

- there is no loss of accessibility
- $n = 4$ and $\dim \Delta_A = 4$
- the bicycle kinematic model is controllable
- the constraints are nonholonomic



A systematic procedure to construct the accessibility distribution

Given the kinematic model of a mobile robot

$$\dot{\mathbf{q}} = G(\mathbf{q})\mathbf{u} = \sum_{i=1}^{m} g_i(\mathbf{q})\, u_i$$

the accessibility distribution Δ_A can be computed in this way:

1. $\Delta_1 = \mathrm{span}\{g_1, \dots, g_m\}$
2. $\Delta_i = \Delta_{i-1} + \mathrm{span}\{[g, v], g \in \Delta_1, v \in \Delta_{i-1}\}$ for $i \geq 2$

This procedure stops when $\Delta_{\kappa+i} = \Delta_\kappa = \Delta_A$, where $\kappa \leq n - m + 1$.

Try to apply this procedure to the underwater spherical robot and to the quadrotor.

Conclusion

Given any mobile robot (ground, underwater/surface, aerial), the kinematic model can be expressed in a general way by the following expression

$$\dot{\mathbf{q}} = G(\mathbf{q})\,\mathbf{u} = \sum_{i=1}^{m} g_i(\mathbf{q})\,u_i$$

and can be derived from the kinematic constraints that limit the motion of the robot.

Analysing the kinematic constraints we can gain more insights on the actual degrees of freedom / motion capabilities of the robot, and, consequently, on the controllability of this dynamic system.

Is the kinematic model accurate enough to be used to design a trajectory tracking controller for a mobile robot?

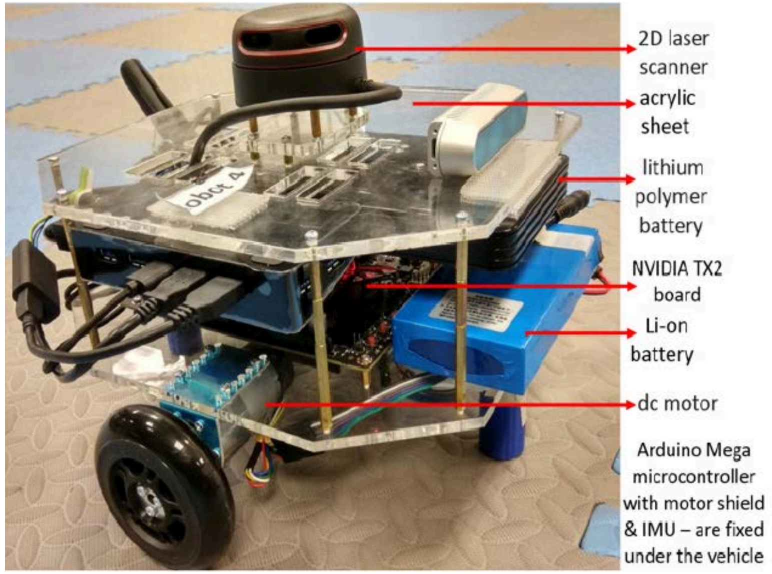
Let's consider the trajectory tracking control of a nonholonomic differential drive mobile robot.

We consider two different trajectory tracking controllers. Both have PI inner loops to control the wheel velocities. The outer loop is designed:

- on the kinematic model
- on the dynamic and kinematic model

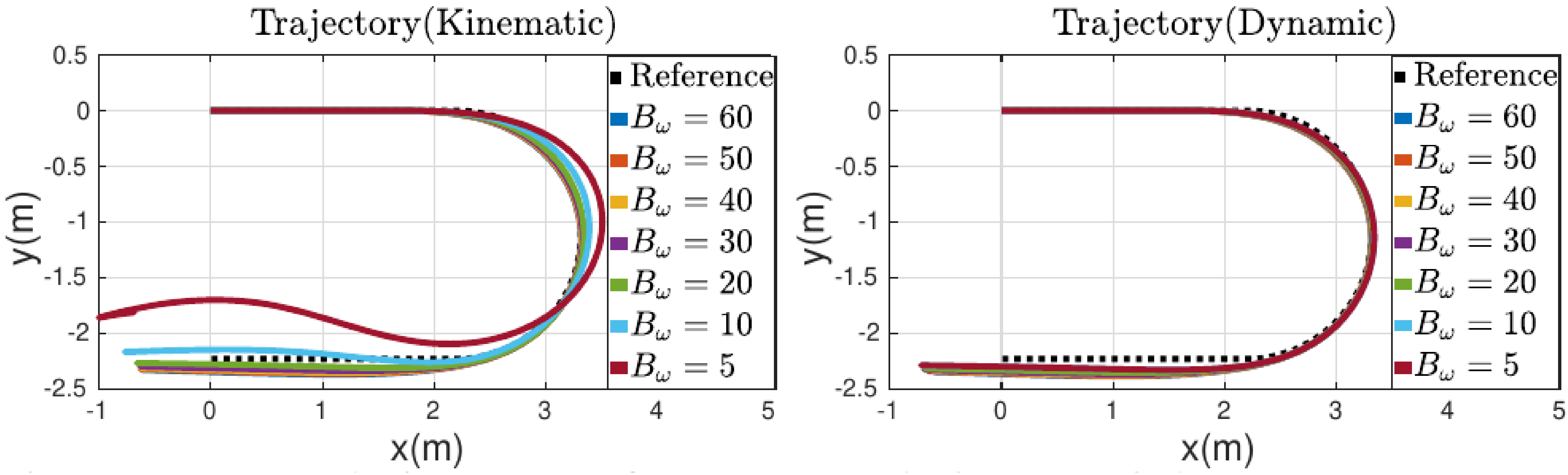
Is there any significant difference in the tracking results?

Do the two controllers exhibit the same performance?

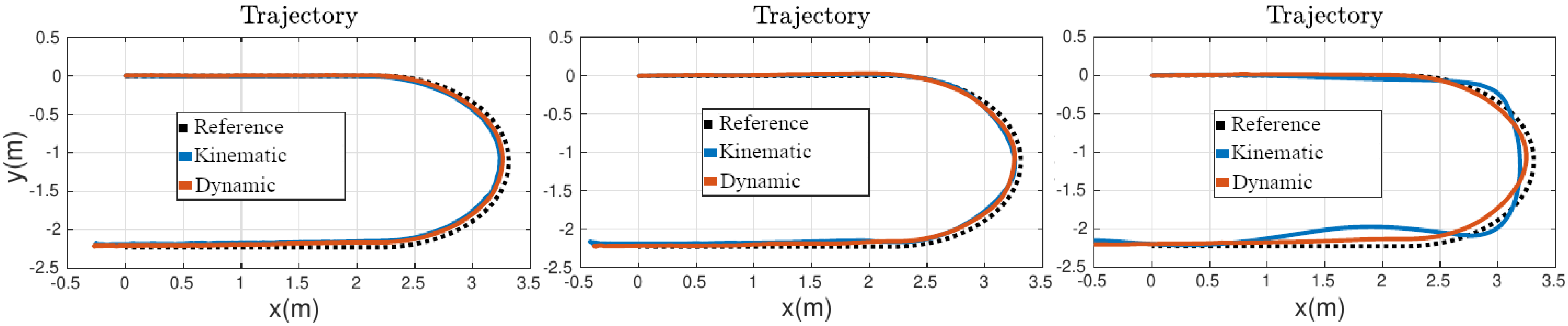


Conclusion

We first consider simulation results...



And then experimental results...



$v = 0.35\, m/s$ $v = 0.55\, m/s$ $v = 0.75\, m/s$

...increasing the velocity the performance with the controller based on the kinematic model decreases!

On the importance of kinematic modelling

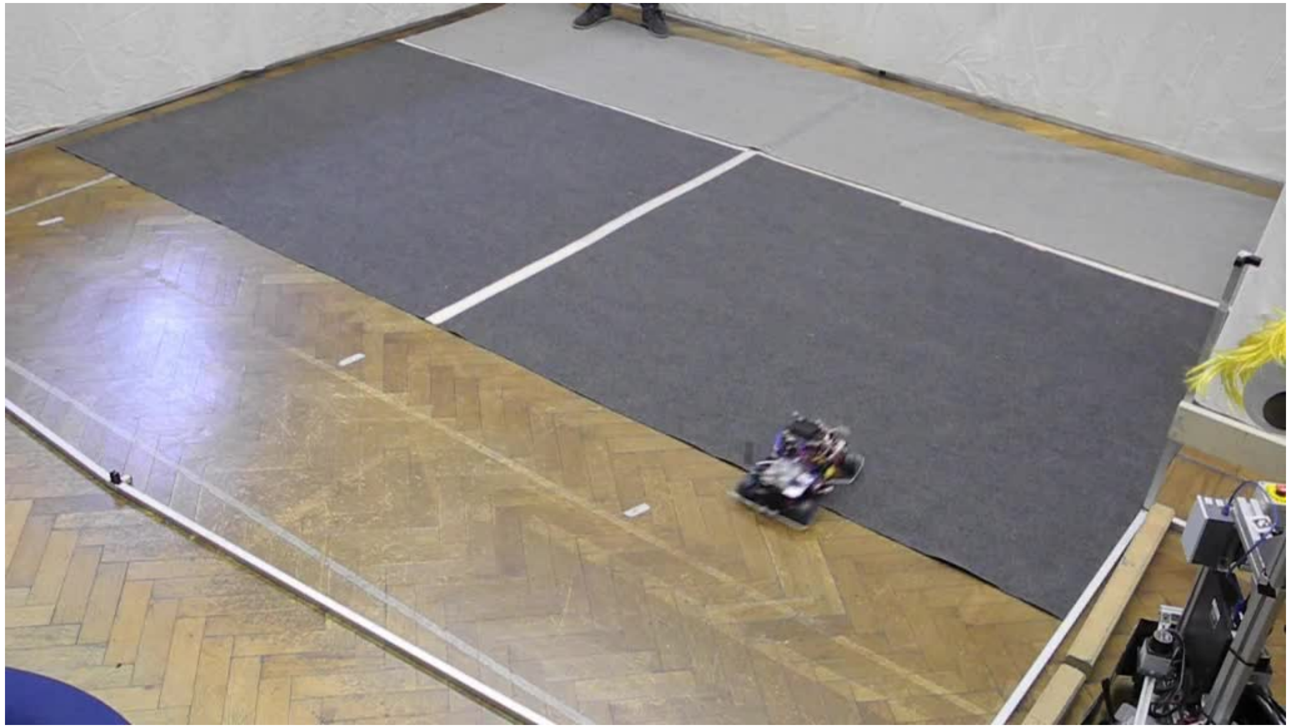
Are kinematic models useful?

Can we use unicycle and bicycle model to represent real mobile robots?

External video

On the importance of kinematic modelling

Is the kinematic model enough?



External video

On the importance of kinematic modelling

Are the unicycle / bicycle approximations realistic for mobile robots and vehicles?

External video

Integrability conditions: remarks

The differential form in three dimensions

$$A(x,y,z)\,\mathrm{d}x + B(x,y,z)\,\mathrm{d}y + C(x,y,z)\,\mathrm{d}z$$

is called an *exact differential* if there exists a scalar function $f(x,y,z)$, such that

$$A(x,y,z) = \frac{\partial f(x,y,z)}{\partial x} \qquad B(x,y,z) = \frac{\partial f(x,y,z)}{\partial y} \qquad C(x,y,z) = \frac{\partial f(x,y,z)}{\partial z}$$

In three dimensions a differential

$$\mathrm{d}f = A(x,y,z)\,\mathrm{d}x + B(x,y,z)\,\mathrm{d}y + C(x,y,z)\,\mathrm{d}z$$

is exact if and only if

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \qquad \frac{\partial A}{\partial z} = \frac{\partial C}{\partial x} \qquad \frac{\partial B}{\partial z} = \frac{\partial C}{\partial y}$$

This conclusion follows from the application of the *Schwarz theorem*.