EXERCISE 1 (Kinematics)

1. Considering a disk rolling without slipping on the horizontal plane, keeping the sagittal plane in the vertical direction. Write the expression of the pure rolling constraint explaining its physical meaning.

2. Show that the pure rolling constraint is a nonholonomic constraint.

3. Is the constraint $\dot{x} + (1 - x)\,\dot{y} + \dot{z} = 0$ holonomic or nonholonomic? Motivate the answer.

EXERCISE 2 (Dynamics)

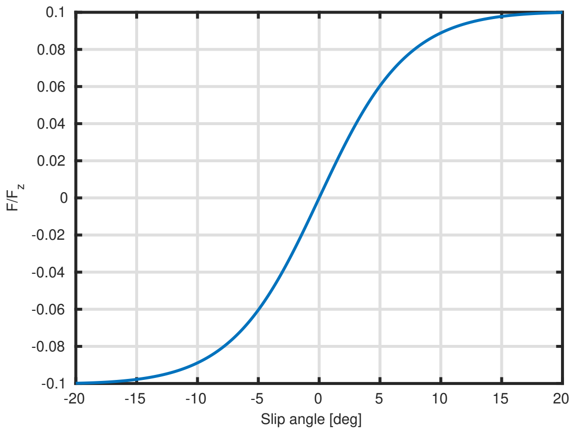
1. Draw the diagram of a bicycle vehicle, showing all the forces acting on it. Write the analytic expression of the force and moment balances.

2. Explain the physical meaning of the front/rear slip angles. How are these angles related to the vehicle dynamics variables (speed, steer, yaw rate, ...)?

3. Assume that the front/rear lateral forces are described by the following linear relations

$$F_{y_F} = -C_{\alpha_F}\alpha_F \qquad F_{y_R} = -C_{\alpha_R}\alpha_R$$

while the experimental lateral force-slip relation is given (for both F_{y_F} and F_{y_R}) by the curve in the following figure



Compute the values of C_{α_F} and C_{α_R}, and explain their relation with the experimental curve.

ESERCIZIO 3 (Planning)

1. Consider a unicycle kinematic model, assuming as flat outputs variables $z_1 = x$ and $z_2 = y$, write the expressions of the flatness transformations.

2. Given as initial state $\mathbf{q}(0) = [x_i, y_i, \theta_i]$ and final state $\mathbf{q}(t_f) = [x_f, y_f, \theta_f]$, write the analytical expression of the trajectory planned using the flatness transformation and explain how this trajectory can be computed.

3. Explain how the previous procedure can be modified in order to introduce a cost function and plan a trajectory that minimizes this cost function as well.

ESERCIZIO 4 (Control)

1. The following control law is used to linearise the dynamics of a unicycle kinematic model

$$v = v_{xP} \cos\theta + v_{yP} \sin\theta$$
$$\omega = \frac{v_{yP} \cos\theta - v_{xP} \sin\theta}{\varepsilon}$$

   where v_{xP} and v_{yP} are the velocities of a point P at a distance ε from the wheel contact point along the sagittal axis of the vehicle. Compute the equations of the closed loop system resulting from the application of the previous law.

2. Design a trajectory tracking controller for a unicycle robot based on the linearising law introduced in Step 1. Draw a block diagram of the entire control system and explain how it can be tuned.

3. Explain why the control system designed in Step 2 cannot be used to regulate the pose of the robot.

Control of Mobile Robots
Prof. Luca Bascetta

EXERCISE 1 (Kinematics)

1. Considering a disk rolling without slipping on the horizontal plane, keeping the sagittal plane in the vertical direction. Write the expression of the pure rolling constraint explaining its physical meaning.

   The pure rolling constraint has the following expression

   $$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

   or, in Pfaffian form

   $$a^T(\mathbf{q})\,\dot{\mathbf{q}} = [\sin\theta \quad -\cos\theta \quad 0]\,\dot{\mathbf{q}} = 0$$

   where $\mathbf{q} = [x \;\; y \;\; \theta]^T$.

   The constraint means that in a disk rolling without slipping the velocity components in the directions perpendicular to the sagittal plane are equal to zero.

2. Show that the pure rolling constraint is a nonholonomic constraint.

   The pure rolling constraint is a nonholonomic constraint because it only locally constraints the set of admissible velocities, but the disk can reach any position in the 2D plane.

   We can show the pure rolling constraint is nonholonomic using the necessary and sufficient condition. If the constraint were holonomic, we should find a function $\alpha(\mathbf{q})$ that satisfies the following equations

   $$\frac{\partial(\alpha(\mathbf{q})\sin\theta)}{\partial y} = -\frac{\partial(\alpha(\mathbf{q})\cos\theta)}{\partial x} \qquad (1)$$

   $$\frac{\partial(\alpha(\mathbf{q})\sin\theta)}{\partial\theta} = 0 \qquad (2)$$

   $$0 = -\frac{\partial(\alpha(\mathbf{q})\cos\theta)}{\partial\theta} \qquad (3)$$

   From equation (1) we get

   $$\sin\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = -\cos\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

   from (2)

   $$\sin\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta} + \alpha(\mathbf{q})\cos\theta = 0 \qquad \Rightarrow \qquad -\cos\theta = \frac{\sin\theta}{\alpha(\mathbf{q})}\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}$$

   and from (3)

   $$-\cos\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta} + \alpha(\mathbf{q})\sin\theta = 0 \qquad \Rightarrow \qquad \sin\theta = \frac{\cos\theta}{\alpha(\mathbf{q})}\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}$$

   Substituting the previous two expressions in equation (1) we get

   $$\frac{\cos\theta}{\cancel{\alpha(\mathbf{q})}}\cancel{\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}}\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = \frac{\sin\theta}{\cancel{\alpha(\mathbf{q})}}\cancel{\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}}\frac{\partial(\alpha(\mathbf{q}))}{\partial x} \qquad \Rightarrow \qquad \cos\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = \sin\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

   or, equivalently

   $$\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = \tan\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

   However, from equation (1) we also get

   $$\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = -\frac{1}{\tan\theta}\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

   These two conditions can be satisfied at the same time if and only if $\alpha(\mathbf{q}) = 0$. We thus conclude that the constraint is nonholonomic.

3. Is the constraint $\dot{x} + (1-x)\,\dot{y} + \dot{z} = 0$ holonomic or nonholonomic? Motivate the answer.

   The constraint can be written in Pfaffian form as follows

$$a^T(\mathbf{q})\,\dot{\mathbf{q}} = [1 \quad (1-x) \quad 1]\,\dot{\mathbf{q}} = 0$$

   where $\mathbf{q} = [x \;\; y \;\; z]^T$.

   We can use again the necessary and sufficient condition. If the constraint were holonomic, we should find a function $\alpha(\mathbf{q})$ that satisfies the following equations

$$\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = \frac{\partial(\alpha(\mathbf{q})(1-x))}{\partial x} \tag{4}$$

$$\frac{\partial(\alpha(\mathbf{q}))}{\partial z} = \frac{\partial(\alpha(\mathbf{q}))}{\partial x} \tag{5}$$

$$\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = \frac{\partial(\alpha(\mathbf{q})(1-x))}{\partial z} \tag{6}$$

   From equation (4) we get

$$\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = (1-x)\frac{\partial(\alpha(\mathbf{q}))}{\partial x} - \alpha(\mathbf{q}) \tag{7}$$

   and from (6)

$$\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = (1-x)\frac{\partial(\alpha(\mathbf{q}))}{\partial z}$$

   Substituting the previous expression and equation (5) in equation (7) we get

$$(1-x)\frac{\partial(\alpha(\mathbf{q}))}{\partial z} = (1-x)\frac{\partial(\alpha(\mathbf{q}))}{\partial z} - \alpha(\mathbf{q})$$

   from which it follows $\alpha(\mathbf{q}) = 0$. We thus conclude that the constraint is nonholonomic.

EXERCISE 2 (Dynamics)

1. Draw the diagram of a bicycle vehicle, showing all the forces acting on it. Write the analytic expression of the force and moment balances.

   See the slides on "Dynamics of mobile robots".

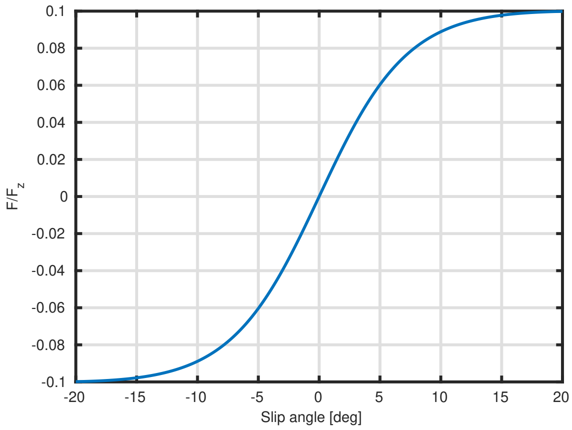
2. Explain the physical meaning of the front/rear slip angles. How are these angles related to the vehicle dynamics variables (speed, steer, yaw rate, ...)?

   Slip angles represent the angles between the wheel sagittal plane and the wheel velocity vector.
   See the slides on "Dynamics of mobile robots" for the relations with the vehicle dynamics variables (assuming usual approximations).

3. Assume that the front/rear lateral forces are described by the following linear relations

$$F_{yF} = -C_{\alpha F}\alpha_F \qquad F_{yR} = -C_{\alpha R}\alpha_R$$

   while the experimental lateral force-slip relation is given (for both F_{yF} and F_{yR}) by the curve in the following figure



Compute the values of C_{α_F} and C_{α_R}, and explain their relation with the experimental curve.

The cornering stiffness C_{α_F} and C_{α_R} represent the slope of the tangent in the origin to the experimental curve. For the curve in the figure the value of the cornering stiffness is approximately $0.8\, N/rad$.

ESERCIZIO 3 (Planning)

1. Consider a unicycle kinematic model, assuming as flat outputs variables $z_1 = x$ and $z_2 = y$, write the expressions of the flatness transformations.

   See the slides on "Control of Mobile Robots".

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   See the slides on "Planning of Mobile Robots".

3. Explain how the previous procedure can be modified in order to introduce a cost function and plan a trajectory that minimizes this cost function as well.

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ESERCIZIO 4 (Control)

1. The following control law is used to linearise the dynamics of a unicycle kinematic model

$$v = v_{xP} \cos\theta + v_{yP} \sin\theta$$
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   where v_{xP} and v_{yP} are the velocities of a point P at a distance ε from the wheel contact point along the sagittal axis of the vehicle. Compute the equations of the closed loop system resulting from the application of the previous law.

   See the slides on "Control of Mobile Robots".

2. Design a trajectory tracking controller for a unicycle robot based on the linearising law introduced in Step 1. Draw a block diagram of the entire control system and explain how it can be tuned.

   See the slides on “Control of Mobile Robots”.

3. Explain why the control system designed in Step 2 cannot be used to regulate the pose of the robot.

   Because applying the feedback linerization the vehicle heading becomes an hidden state, and it is thus no more controllable.