



Automatic Control

Root locus

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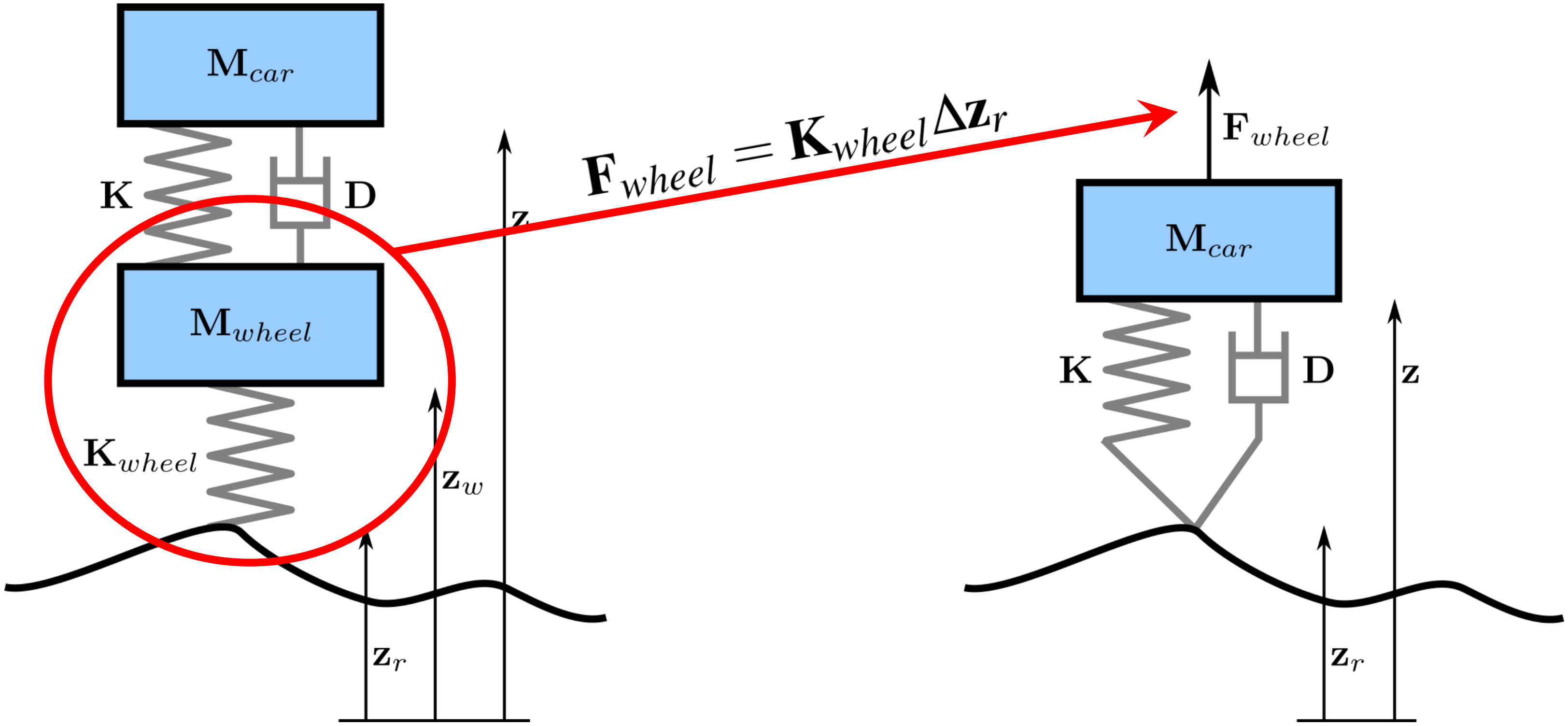
Motivations (I)

Consider the following problems:

- determine the damping coefficient of a passive damper
- determine the damping coefficient of a switchable damper

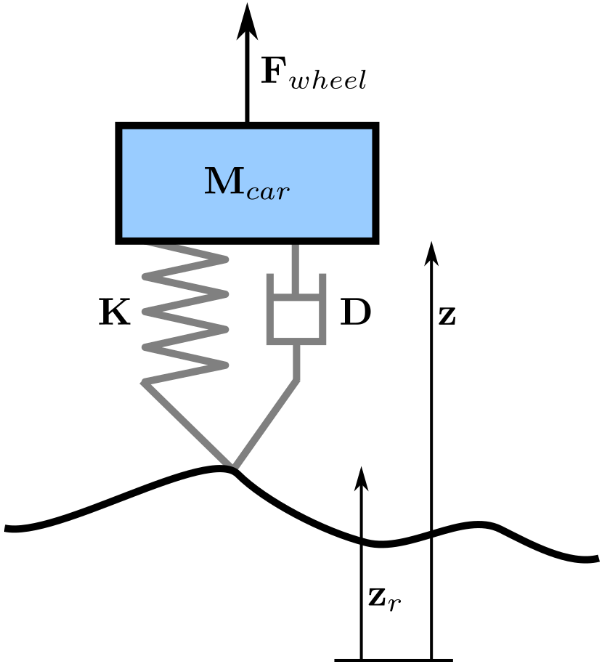
in order to enforce a desired behavior of the suspension system



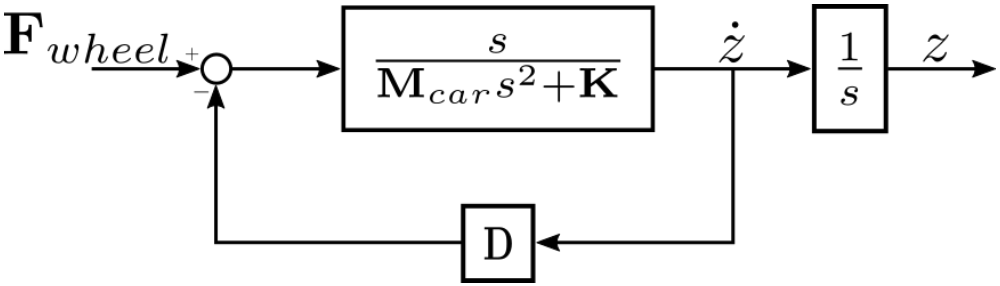
Motivations (II)

We can now write the differential equation that describes the suspension system as follows

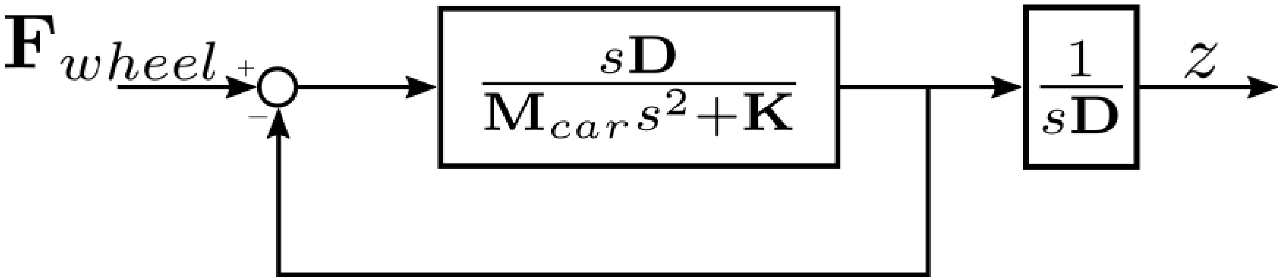
$$\mathbf{M}_{car}\ddot{z} + \mathbf{D}\dot{z} + \mathbf{K}z = \mathbf{F}_{wheel}$$



The same relation can be expressed using a closed-loop system

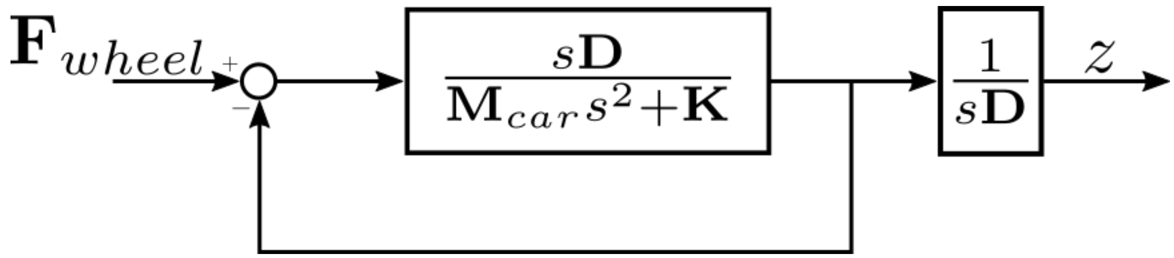


That can be further elaborated as follows

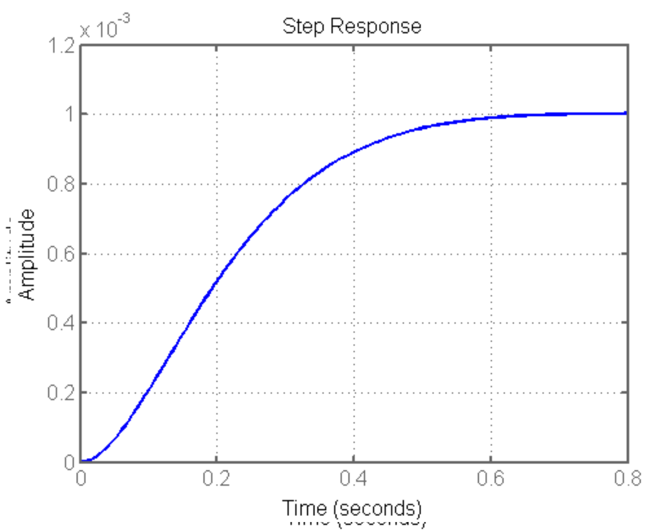


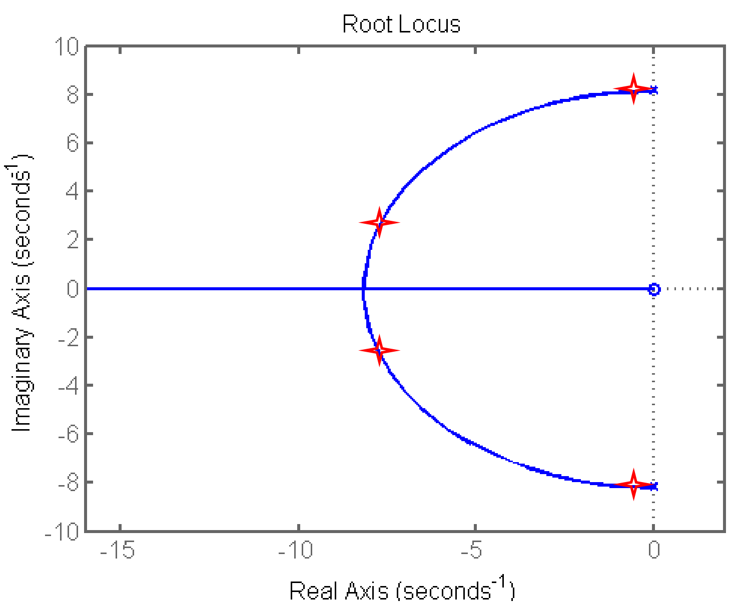
Motivations (III)

The original problem (changing the damping coefficient in order to enforce a desired behavior of the suspension system) can be now reformulated as



Determine the positions in the complex plane of the poles of the closed-loop system for any positive value of ***D****.*





Goals

Given a loop transfer function

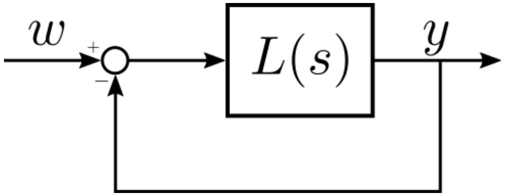
$$L(s) = \rho \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

ρ gain

$-p_i, -z_i$ poles and zeros

we aim at studying the poles of the closed-loop system or, equivalently, the roots of the characteristic equation

$$1 + L(s) = 0$$

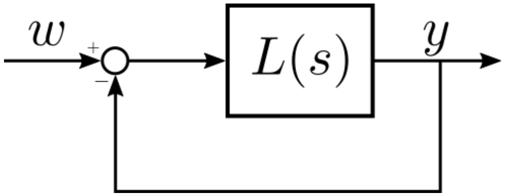


and, in particular, how these roots move in the complex plain when the value of the gain changes.

Root locus (I)

Given a loop transfer function

$$L(s) = \rho \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$



we define *root locus* the curve in the complex plane generated by the roots of the characteristic equation for different values of the real parameter ρ in the range $[-\infty, 0)(0, +\infty]$

- $\rho \in (0, +\infty]$, *"direct" root locus (negative feedback)*
- $\rho \in [-\infty, 0)$, *"inverse" root locus (positive feedback)*

Note that, for $\rho = 0$ we have no feedback, and thus the closed-loop poles correspond to the poles of $L(s)$.

Assuming that the loop transfer function can be expressed as

$$L(s) = \rho \frac{N(s)}{D(s)}$$

the characteristic equation is equivalent to the following complex equation

$$\frac{N(s)}{D(s)} = -\frac{1}{\rho}$$

that can be expressed by way of two real equations

$$\frac{|N(s)|}{|D(s)|} = \frac{1}{|\rho|}$$

$$\angle N(s) - \angle D(s) = 180^\circ - \angle \rho = \begin{cases} (2k+1)180^\circ & \rho > 0 \\ 2k180^\circ & \rho < 0 \end{cases}$$

Given $\bar{s} \in \mathfrak{C}$, belonging to the locus, this equation determines the value of ρ for which a closed-loop pole is located at $\bar{s}$

This equation determines the shape of the root locus

Sketching the root locus (I)

How to plot the root locus of $L(s) = \frac{\rho}{s(s+3)(s+5)}$?

Using MATLAB…

```
» L = tf(1,conv([1 0],conv([1 3],[1 5])));
» figure, rlocus(L)
```

… or introducing a set of simple rules that allow to sketch the root locus using minimal calculation, so as to have an intuitive insight into the behavior of the closed-loop system.

Sketching the root locus (II)

Consider again a loop transfer function $L(s)$ where

- m, is the degree of the numerator $N(s)$
- n, is the degree of the denominator $D(s)$
- $\nu = n - m > 0$, is the relative degree

For a given ρ, we define *centroid* or *center of gravity* the sum of closed-loop poles divided by n, as follows

$$\text{centroid} = -\frac{1}{n}\sum_{i=1}^{n} p_i$$

Sketching the root locus (III)

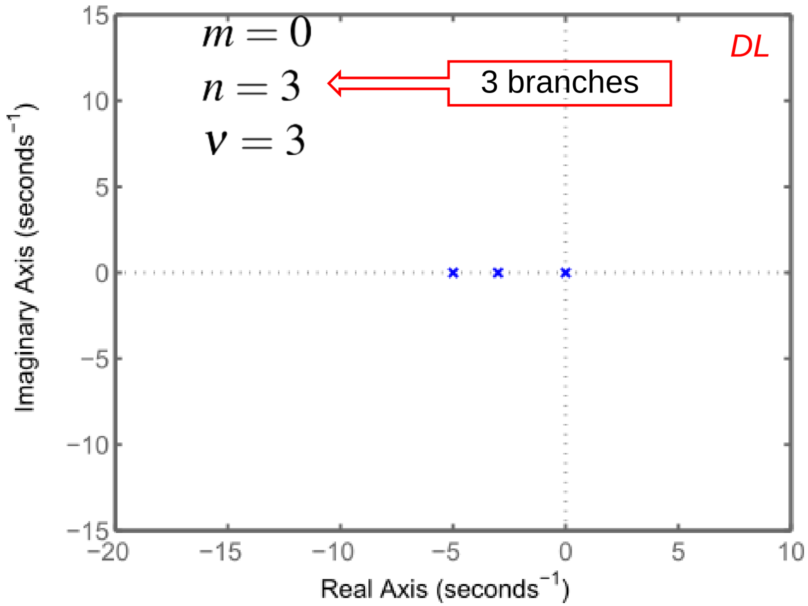
Rule 1. The root locus has $2n$ branches (closed-loop poles migrating in the complex plane): n belong to the "direct" locus, n to the "inverse" one.

Rule 2. The root locus is symmetric with respect to the real axis.

Rule 3. Branches begin ($|\rho| \to 0$) at poles of $L(s)$.

Rule 4. When $|\rho| \to \infty$, m branches end at zeros of $L(s)$, ν branches go to infinity approaching an asymptote.

$$L(s) = \frac{\rho}{s(s+3)(s+5)}$$



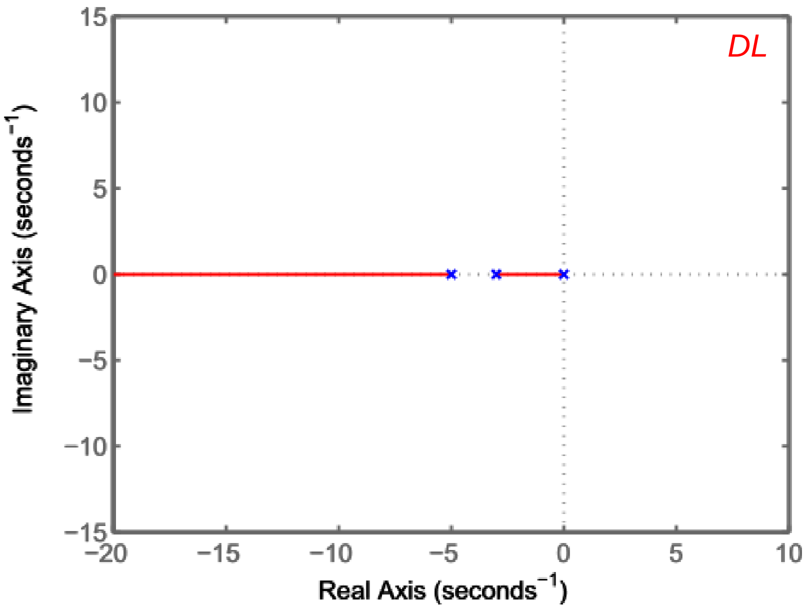
Sketching the root locus (IV)

Rule 5. The root locus lies on the real axis to the left of an

- odd number (direct locus)
- even number (inverse locus)

of singularities (poles and zeroes).

$$L(s) = \frac{\rho}{s(s+3)(s+5)}$$



Rule 6. Asymptotes intersect at a point on the real axis, whose coordinate is given by

$$x_a = \frac{1}{\nu}\left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i\right)$$

They form an angle with the real axis, whose value is given by

$$\theta_a = \begin{cases} \frac{180^\circ + k\cdot 360^\circ}{\nu} & k = 0,1,\dots,\nu-1 \quad \text{direct locus} \\ \frac{k\cdot 360^\circ}{\nu} & k = 0,1,\dots,\nu-1 \quad \text{inverse locus} \end{cases}$$

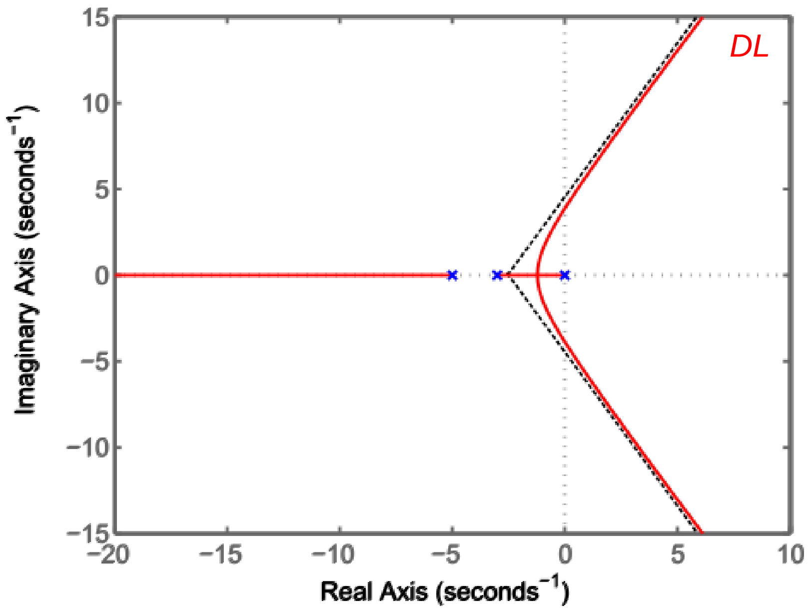
Sketching the root locus (V)

Rule 5. The root locus lies on the real axis to the left of an

- odd number (direct locus)
- even number (inverse locus)

of singularities (poles and zeroes).

$$L(s) = \frac{\rho}{s(s+3)(s+5)}$$



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$$\theta_a = \begin{cases} \frac{180^\circ + k\cdot 360^\circ}{3} = 60^\circ, 180^\circ, 300^\circ & DL \\ \frac{k\cdot 360^\circ}{3} = 0^\circ, 120^\circ, 240^\circ & IL \end{cases}$$

$$x_a = -\frac{1}{3}(0+3+5) = -\frac{8}{3}$$

Sketching the root locus (VI)

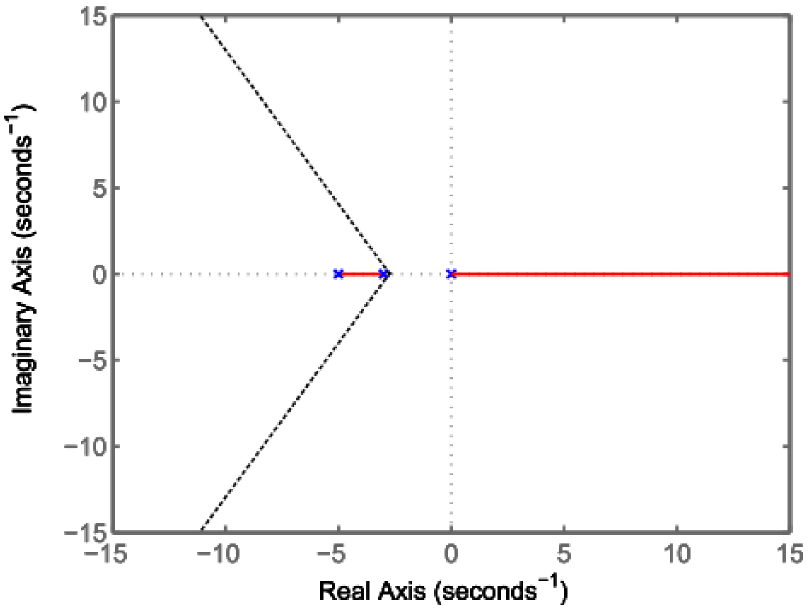
Consider again the transfer function $L(s) = \frac{\rho}{s(s+3)(s+5)}$, we would like to sketch the inverse locus.

From the previous analysis it follows that

$$m = 0 \qquad n = 3 \qquad \nu = 3$$

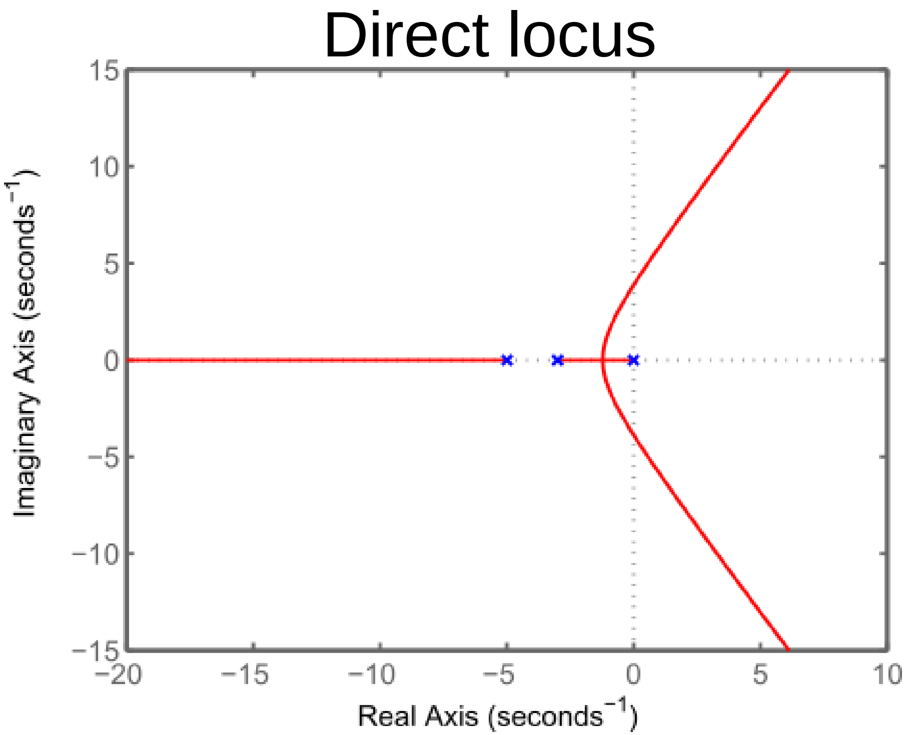
$$x_a = -\frac{1}{3}(0+3+5) = -\frac{8}{3}$$

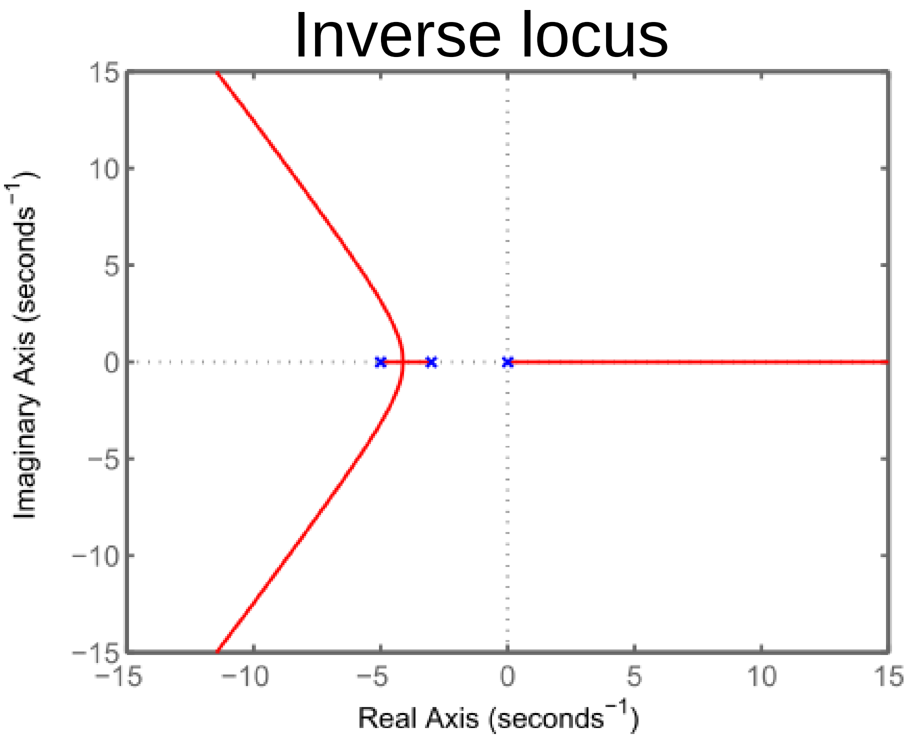
$$\theta_a = \begin{cases} \frac{180^\circ + k\cdot 360^\circ}{3} = 60^\circ, 180^\circ, 300^\circ & DL \\ \frac{k\cdot 360^\circ}{3} = 0^\circ, 120^\circ, 240^\circ & IL \end{cases}$$

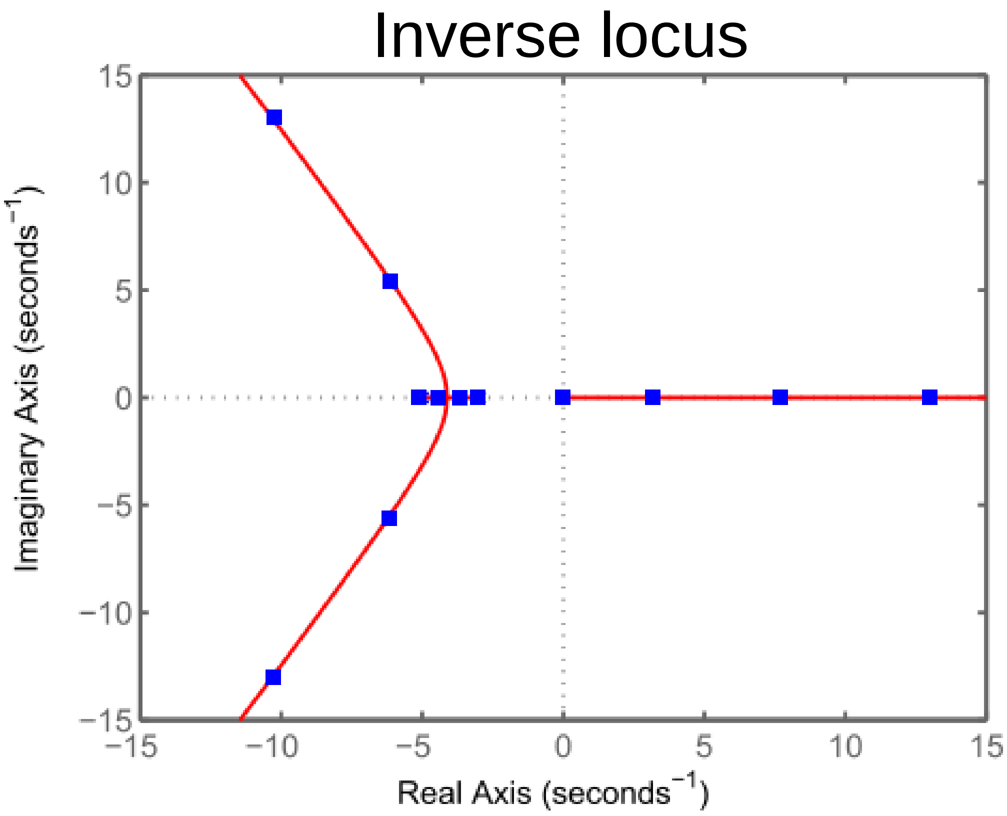


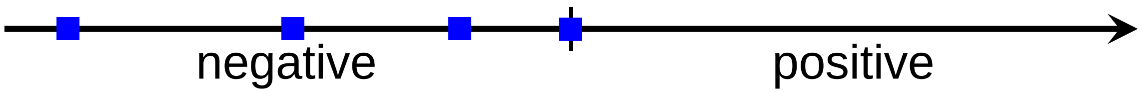
Stability analysis (I)

Now that we have sketched the direct and inverse root loci, we are interested to study the stability of the closed-loop system, i.e., to determine the values of ρ for which the closed-loop poles lie in the open left half plane.



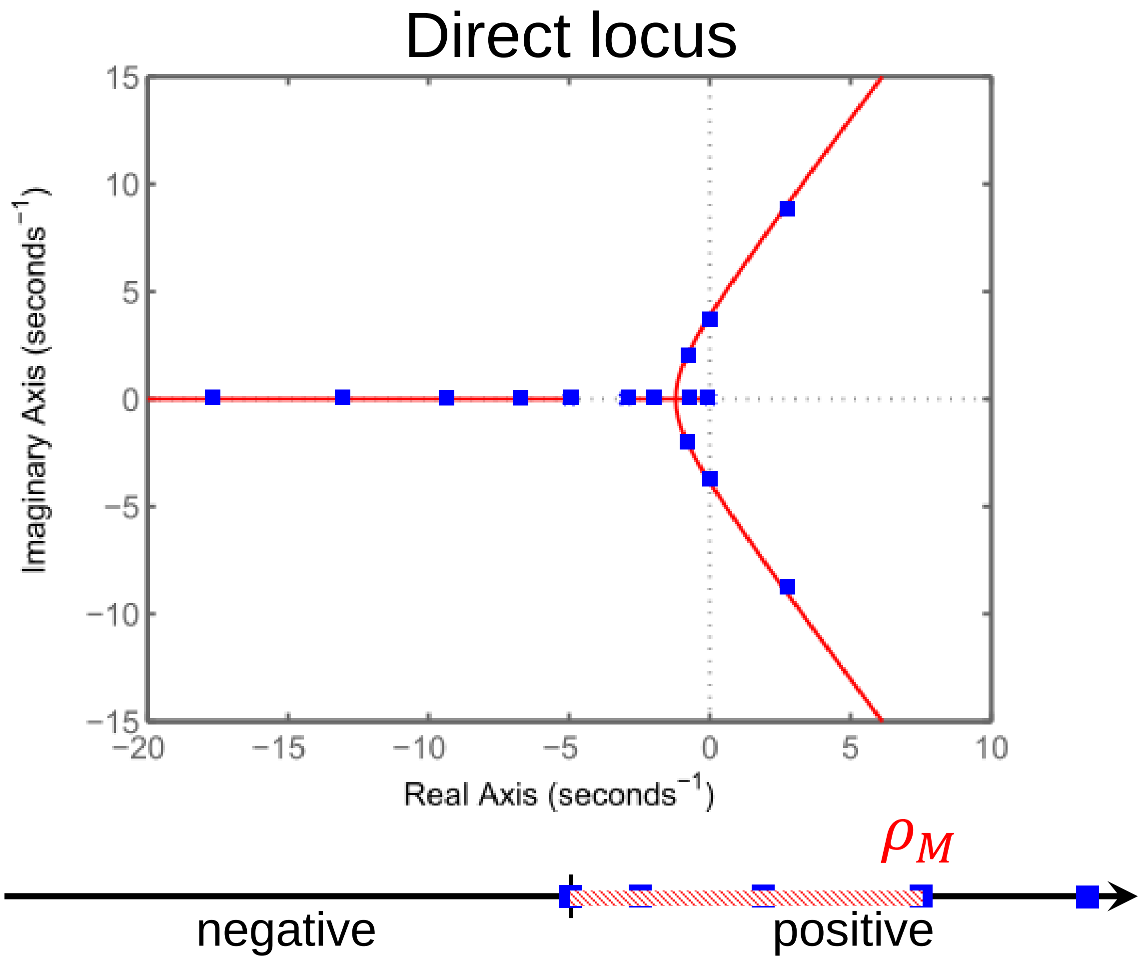






There are no negative values of ρ for which the closed-loop system is asymptotically stable.

Stability analysis (III)



For $\rho < \rho_M$ the closed-loop system is asymptotically stable.

How to find the value of ρ_M?

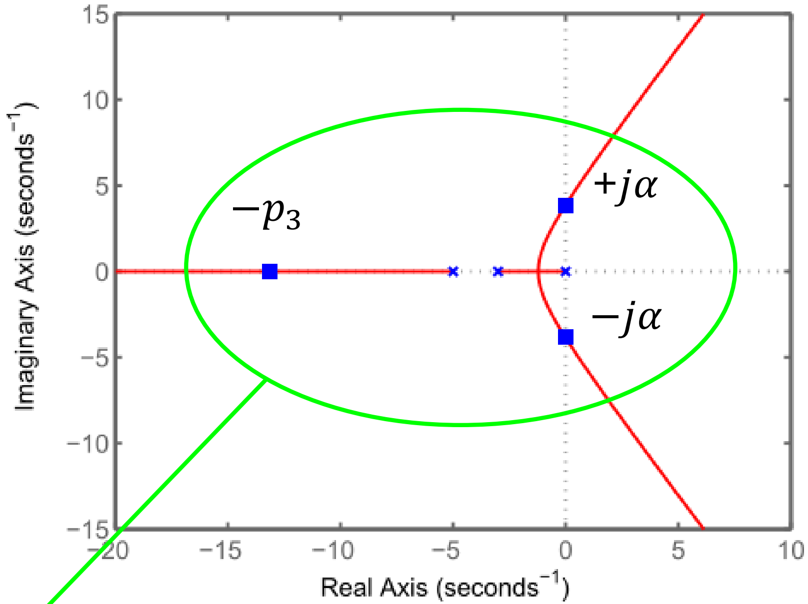
Stability analysis (IV)

How to find the value of ρ_M? …we need two more rules!

Rule 7. When $\nu \geq 2$, the center of gravity is constant (i.e., it anymore depends on ρ) and lies at a point on the real axis, whose coordinate is given by

$$\text{centroid} = -\frac{1}{n}\sum_{i=1}^{n} p_i$$

$$L(s) = \frac{\rho}{s(s+3)(s+5)}$$



$$\text{centroid} = -\frac{0+3+5}{3} = -\frac{8}{3}$$

Thanks to rule 7 we know that, when two of the poles are located on the imaginary axis, the third one, that is on the real axis, is located at

$$-\frac{1}{3}(j\alpha - j\alpha + p_3) = -\frac{8}{3} \Rightarrow -p_3 = -8$$

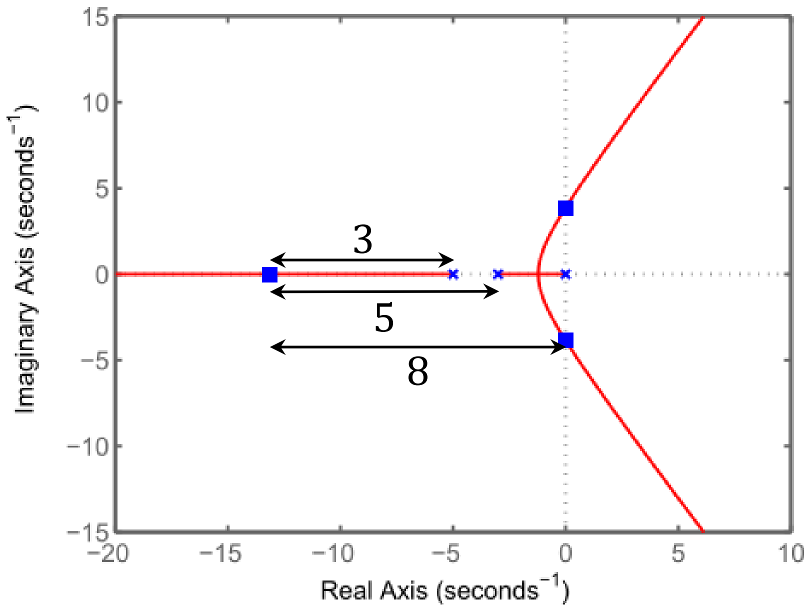
How to find the value of ρ_M? …it is the value of the gain for which the third pole is located at -8.

The problem is now: *how to find the value of the gain for which one of the closed-loop poles is at a specific point on the locus?*

$$L(s) = \frac{\rho}{s(s+3)(s+5)}$$

Rule 8. At any point $\bar{s}$ on the locus, the absolute value of the gain can be calculated as the product of distances from the point to the poles divided by the product of distances from the point to the zeroes (if there are no zeroes, the denominator is 1)

$$|\rho| = \frac{\prod_{i=1}^{n} |\bar{s}+p_i|}{\prod_{i=1}^{m} |\bar{s}+z_i|}$$

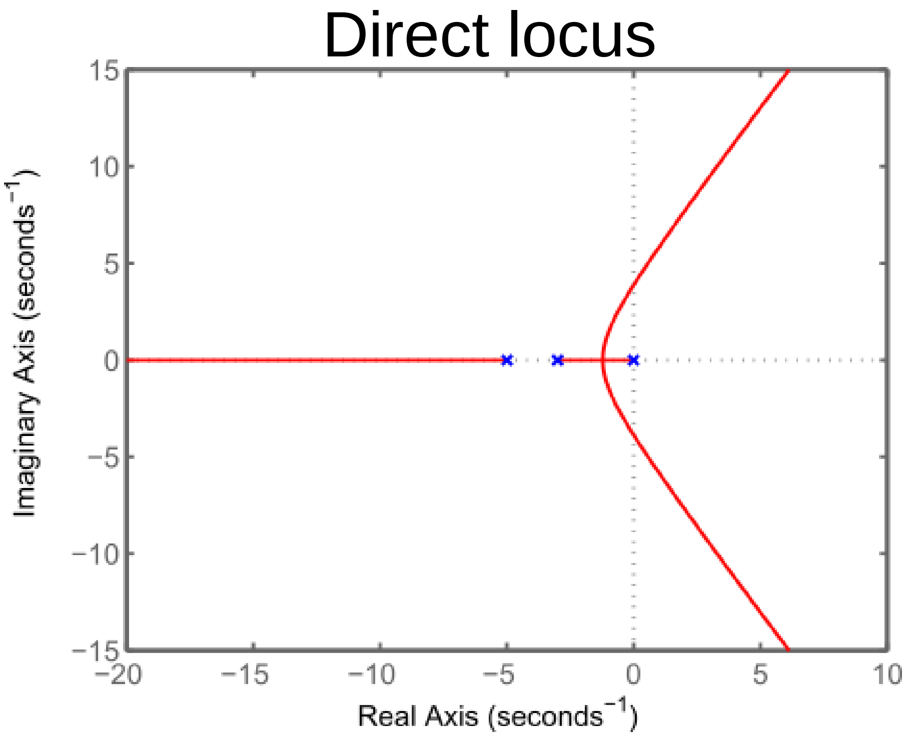


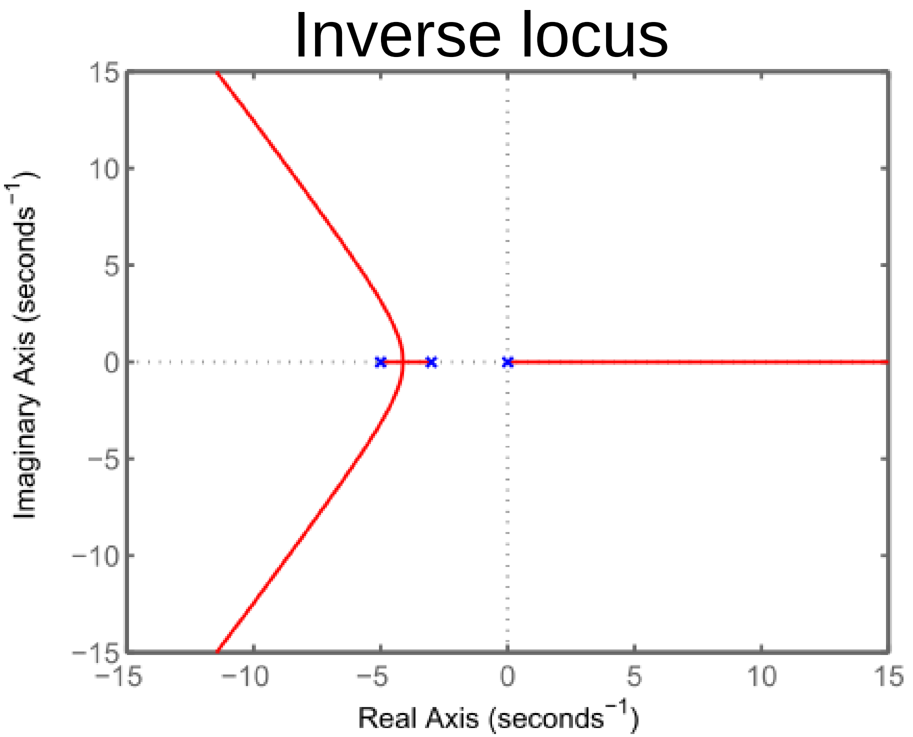
$$\bar{s} = -8 \Rightarrow |\rho| = \frac{3 \cdot 5 \cdot 8}{1} = 120$$

Stability analysis (VI)

The conclusion of the stability analysis is that the closed-loop system is asymptotically stable for

$$0 < \rho < 120$$





Caveat!

There are 3 other rules that allow to refine the sketch of the root locus… we will not see them!

- If you need a refined sketch you can use Matlab!

Sometimes there are more sketches that satisfy the rules

- If you need to disambiguate this situation you can use Matlab!

Sometimes the closed-loop poles configuration is too complex (e.g. two poles on the imaginary axis and two on the real axis) and the centroid rule is not sufficient to perform the stability analysis.

- A numerical analysis can be performed using Matlab!

Transient response design and stabilization (I)

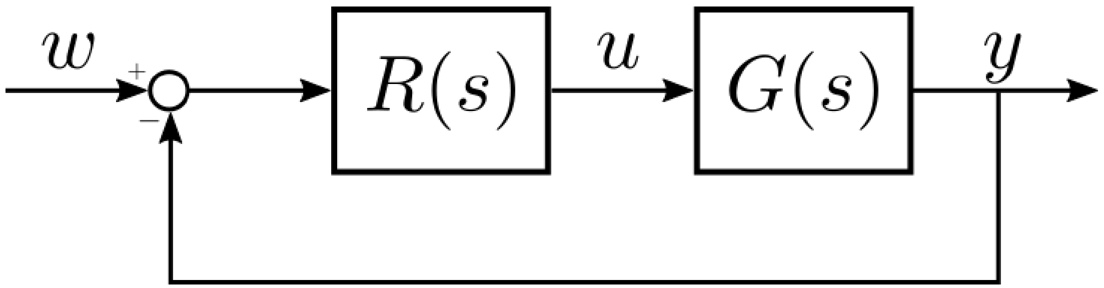
The root locus can be used to

- shape the closed-loop transient response
- stabilize the closed-loop system (e.g., starting from an unstable loop transfer function)

The root locus cannot be used to arbitrarily determine the places of the closed-loop poles, e.g., two points of the complex plane that do not belong to the locus...

- we will see another tool to solve this problem

Consider the following control system



where $G(s) = \frac{1}{(s+1)(s-2)}$.

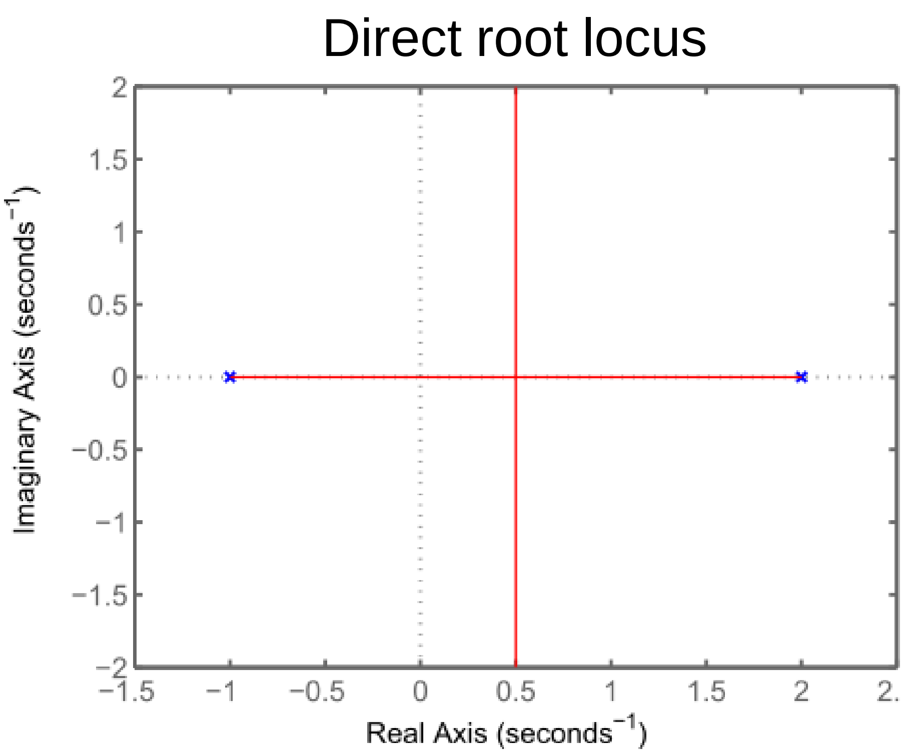
Design the controller $R(s)$ so that the closed-loop system is asymptotically stable and has two real closed-loop poles placed at -2.

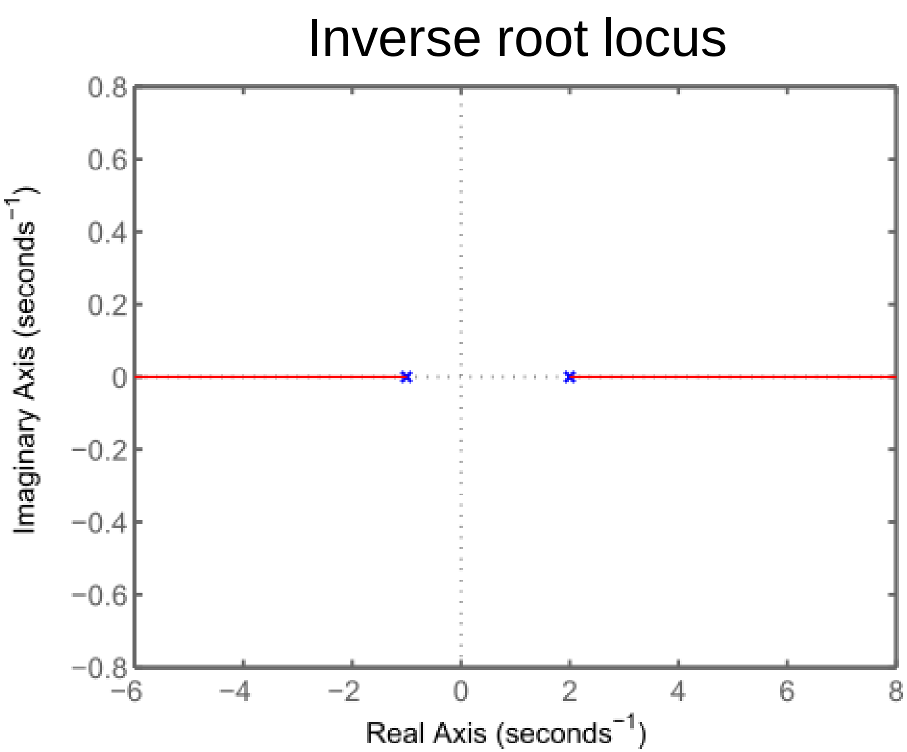
Let's start considering a simple algebraic regulator

$$R(s) = \rho$$

the loop transfer function becomes

$$L(s) = \frac{\rho}{(s+1)(s-2)}$$

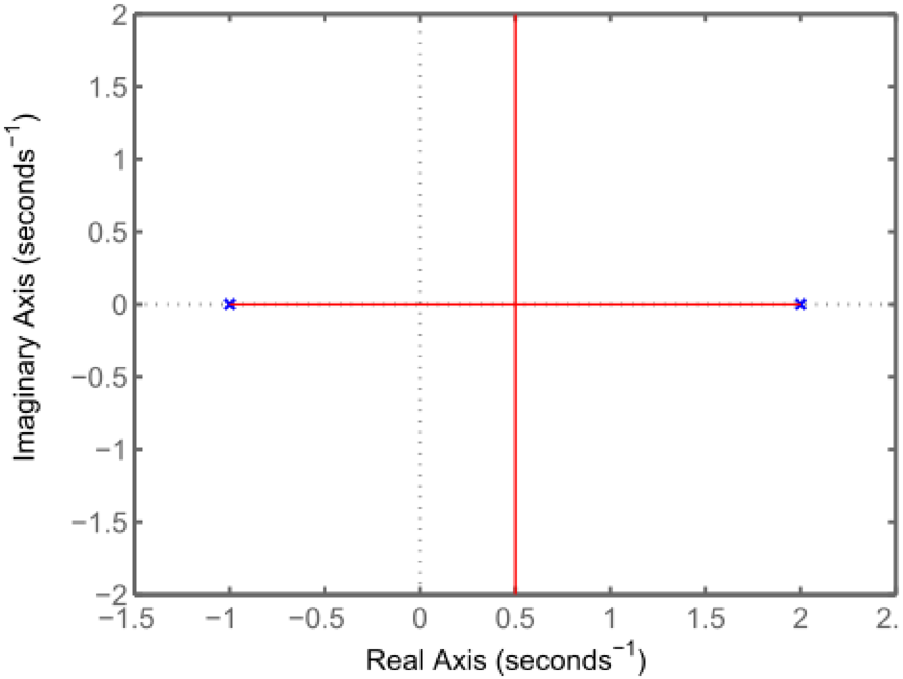




With an algebraic regulator, for any value of ρ the closed-loop system is not asymptotically stable

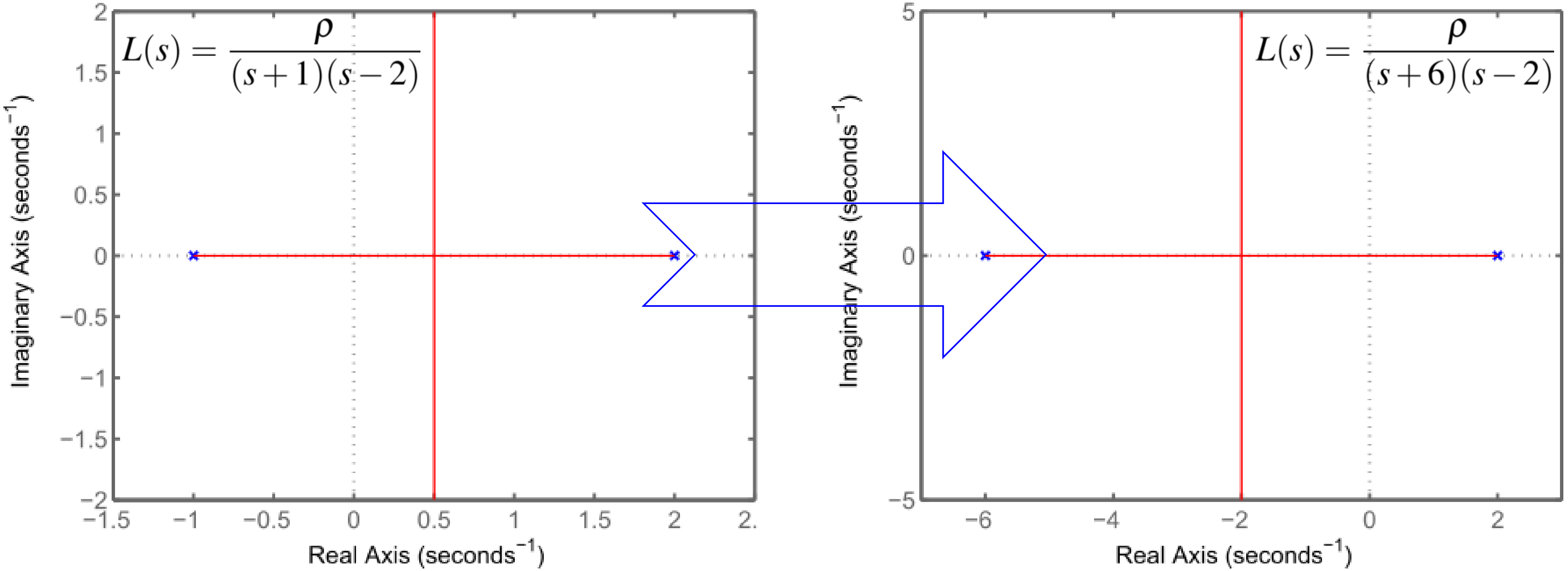
- we have to introduce a more complex regulator

Let’s consider the direct root locus.



- to stabilize the closed-loop system
  - we have to move the vertical asymptote to the left half plane
- to place the closed-loop poles at -2
  - the asymptote must cross the real axis at -2

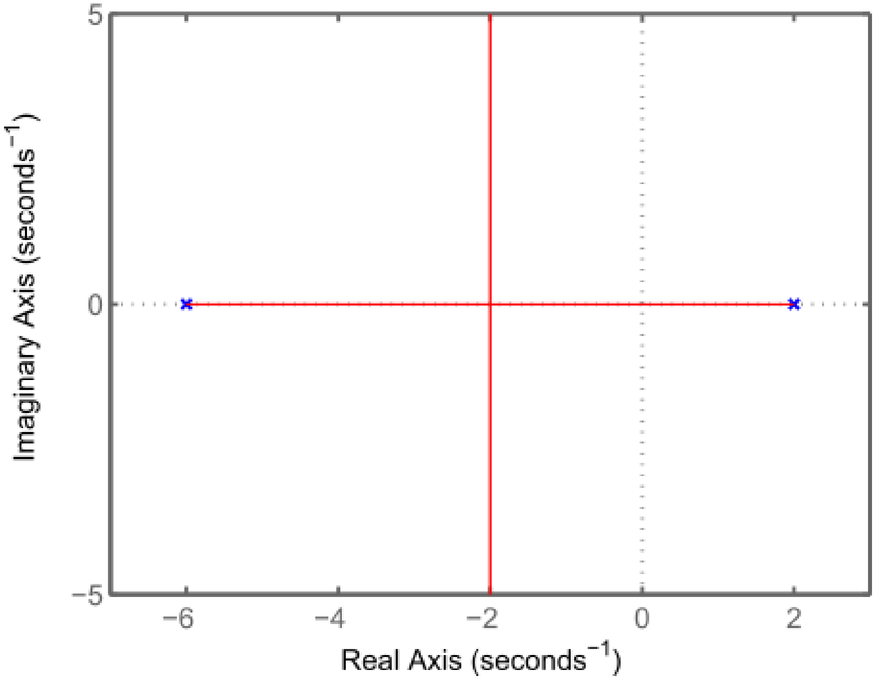
We *cannot remove* the pole in the right half plane, but we *can remove* the pole in the left half plane and substitue it with a pole at -6.



The controller that achieves this result is

$$R(s) = \rho \frac{s+1}{s+6}$$

Now there are values of ρ for which the closed-loop system is asymptotically stable and, in particular, there is a value that gives rise to two real closed-loop poles placed at -2.



Let's compute the value $\bar{\rho}$ that places the poles at -2

$$\bar{\rho} = 4 \cdot 4 = 16$$