Automatic Control
Exercise 5: Root locus and pole placement design
Prof. Luca Bascetta

Exercise 1

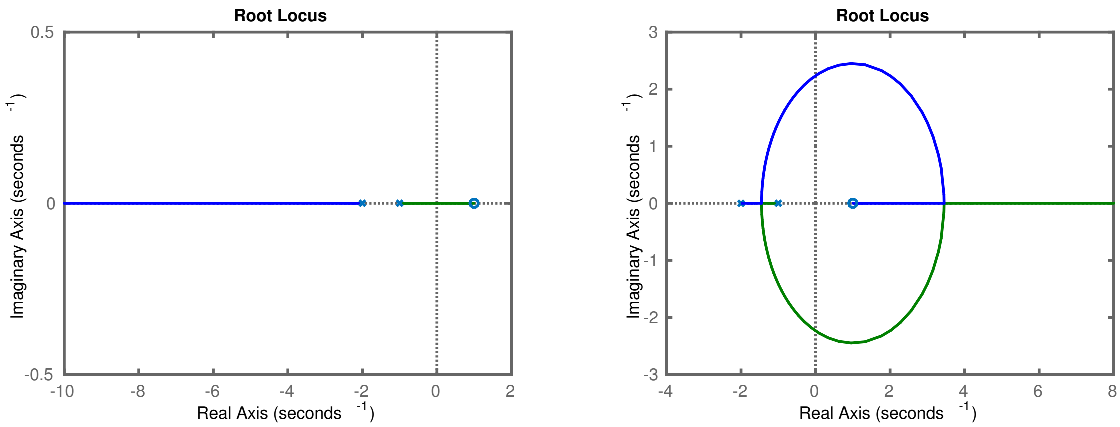
Sketch the direct and inverse root loci of the following loop transfer function

$$L(s) = \rho \frac{s-1}{(s+1)(s+2)}$$

Verify the destabilizing effect of the zero in the right half plane for $|\rho| \to +\infty$.

Solution

The direct and inverse loci are shown in Fig. 1.



(a) direct locus

(b) inverse locus

Figure 1: Direct and inverse root loci.

The asymptotes are characterised by the following angles

$$\theta_a = \begin{cases} \dfrac{180^\circ + h \cdot 360^\circ}{1} = 180^\circ \\ \dfrac{h \cdot 360^\circ}{1} = 0^\circ \end{cases}$$

In the direct locus the pole starting at -1 enters the right half plane as $\rho \to +\infty$ attracted by the right half plane zero. In the inverse locus the two poles starting at -1 and -2 enter the right half plane as $\rho \to -\infty$ attracted by the right half plane zero.

Exercise 2

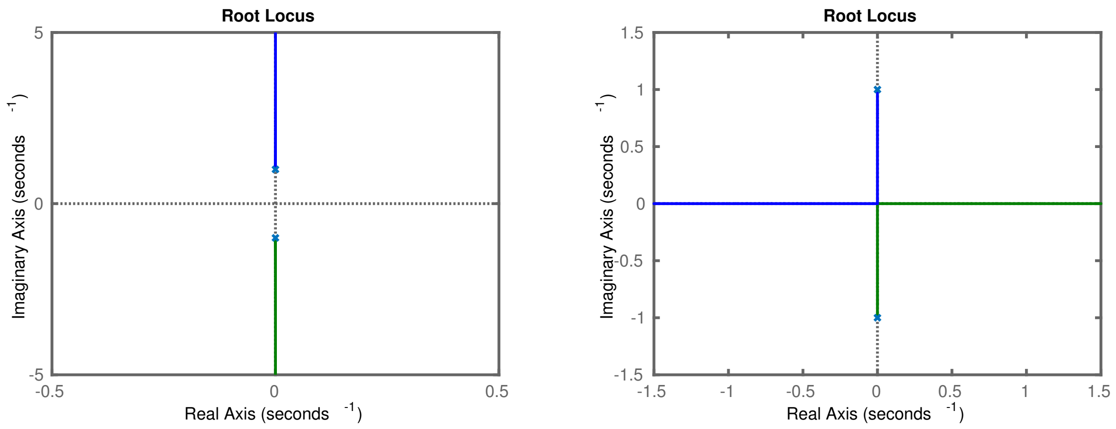
Sketch the direct and inverse root loci of the following loop transfer function

$$L(s) = \frac{\rho}{s^2 + \bar{\omega}^2}$$

Show that there are no values of ρ for which the closed-loop system is asymptotically stable.

Solution

The direct and inverse loci for $\bar{\omega} = 1$ are shown in Fig. 2.



(a) direct locus (b) inverse locus

Figure 2: Direct and inverse root loci.

The asymptotes are characterised by the following angles

$$\theta_a = \begin{cases} \dfrac{180^\circ + h \cdot 360^\circ}{2} = 90^\circ, 270^\circ \\ \dfrac{h \cdot 360^\circ}{2} = 0^\circ, 180^\circ \end{cases}$$

and cross the real axis at

$$x_a = \frac{-(j\bar{\omega} - j\bar{\omega})}{2} = 0$$

In the direct locus there are always two poles on the imaginary axes, in the inverse locus there are always two poles on the imaginary axes or one pole in the right half plane. The closed-loop system is thus never asymptotically stable.

Exercise 3

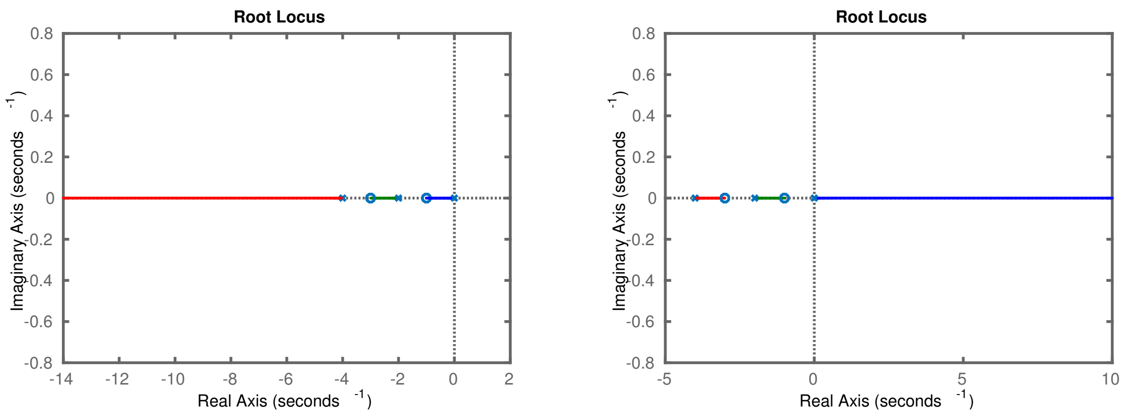
Sketch the direct and inverse root loci of the following loop transfer function

$$L(s) = \rho \frac{(s+1)(s+3)}{s(s+2)(s+4)}$$

Find the values of ρ for which the closed-loop system is asymptotically stable.

Solution

The direct and inverse loci are shown in Fig. 3.



(a) direct locus (b) inverse locus

Figure 3: Direct and inverse root loci.

The asymptotes are characterised by the following angles

$$\theta_a = \begin{cases} \dfrac{180^\circ + h \cdot 360^\circ}{1} = 180^\circ \\ \dfrac{h \cdot 360^\circ}{1} = 0^\circ \end{cases}$$

In the direct locus all the poles are always in the left half plane, in the inverse locus there is always one pole in the right half plane. The closed-loop system is thus asymptotically stable for $\rho > 0$.

Exercise 4

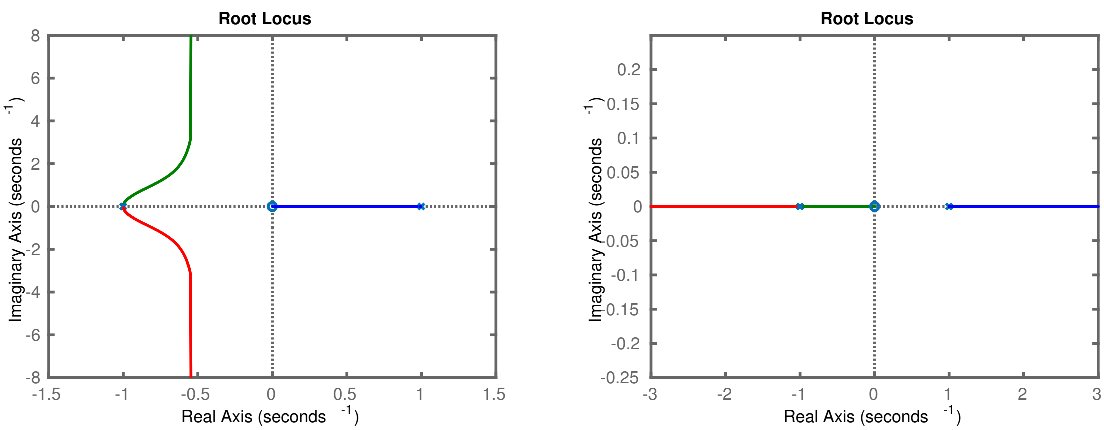
Sketch the direct and inverse root loci of the following loop transfer function

$$L(s) = \rho \frac{s}{(s+1)^2(s-1)}$$

Find the values of ρ for which the closed-loop system is asymptotically stable.

Solution

The direct and inverse loci are shown in Fig. 4.



(a) direct locus

(b) inverse locus

Figure 4: Direct and inverse root loci.

The asymptotes are characterised by the following angles

$$\theta_a = \begin{cases} \dfrac{180^\circ + h \cdot 360^\circ}{2} = 90^\circ, 270^\circ \\ \dfrac{h \cdot 360^\circ}{2} = 0^\circ, 180^\circ \end{cases}$$

and cross the real axis at

$$x_a = \frac{0 - (1 + 1 - 1)}{2} = -\frac{1}{2}$$

In the direct and inverse locus there is always a pole in the right half plane. The closed-loop system is thus never asymptotically stable.

Exercise 5

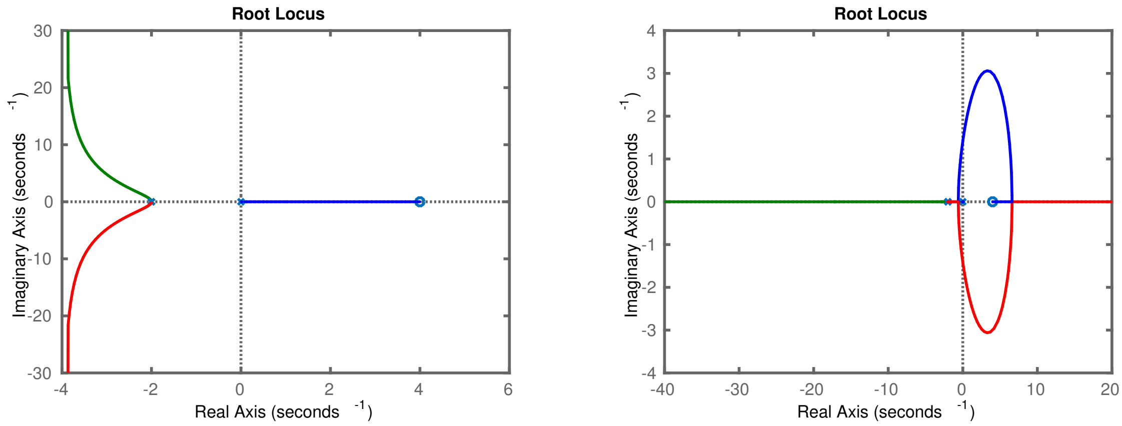
Sketch the direct and inverse root loci of the following loop transfer function

$$L(s) = \rho \frac{s-4}{s(s+2)^2}$$

Find the values of ρ for which the closed-loop system is asymptotically stable.

Solution

The direct and inverse loci are shown in Fig. 5.



(a) direct locus (b) inverse locus

Figure 5: Direct and inverse root loci.

The asymptotes are characterised by the following angles

$$\theta_a = \begin{cases} \dfrac{180^\circ + h \cdot 360^\circ}{2} = 90^\circ, 270^\circ \\ \dfrac{h \cdot 360^\circ}{2} = 0^\circ, 180^\circ \end{cases}$$

and cross the real axis at

$$x_a = \frac{-4 - (0 + 2 + 2)}{2} = -4$$

In the direct locus there is always a pole in the right half plane.
In the inverse root locus there are two complex poles that as ρ increases cross the imaginary axis at an unknown position. The value of ρ for which the poles are on the imaginary axis can be computed using the centroid rule. The centroid is given by

$$\bar{s} = -2 - 2 = -4$$

and the value of ρ for which there is a pole at $\bar{s}$ is given by

$$|\rho_m| = \frac{2 \cdot 2 \cdot 4}{8} = 2$$

The closed-loop system is thus asymptotically stable for $-2 < \rho < 0$.

Exercise 6

Sketch the direct and inverse root loci of the following loop transfer function

$$L(s) = \rho \frac{0.5(s-3)}{(1+s)(1+0.5s)(s+3)}$$

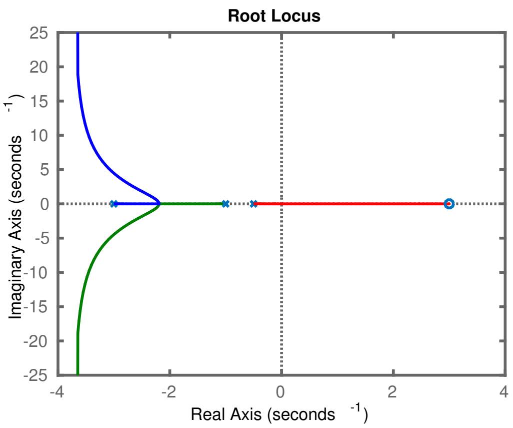
Find the values of ρ for which the closed-loop system is asymptotically stable.

Solution

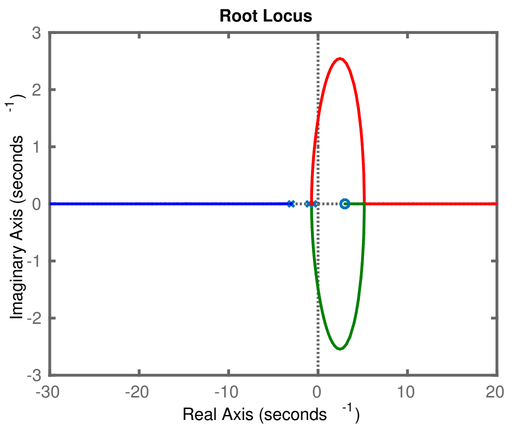
The loop transfer function can be rewritten as

$$L(s) = \rho \frac{(s-3)}{(s+1)(s+2)(s+3)}$$

The direct and inverse loci are shown in Fig. 6.



(a) direct locus



(b) inverse locus

Figure 6: Direct and inverse root loci.

The asymptotes are characterised by the following angles

$$\theta_a = \begin{cases} \dfrac{180^\circ + h \cdot 360^\circ}{2} = 90^\circ, 270^\circ \\ \dfrac{h \cdot 360^\circ}{2} = 0^\circ, 180^\circ \end{cases}$$

and cross the real axis at

$$x_a = \frac{-3 - (1 + 2 + 3)}{2} = -\frac{9}{2}$$

In the direct locus there is a pole that as ρ increases crosses the imaginary axis at $s = 0$. The value of ρ for which the pole is at $s = 0$ is given by

$$\rho_M = \frac{1 \cdot 2 \cdot 3}{3} = 2$$

In the inverse root locus there are two complex poles that as ρ increases cross the imaginary axis at an unknown position. The value of ρ for which the poles are on the imaginary axis can be computed using the centroid rule. The centroid is given by

$$\bar{s} = -3 - 2 - 1 = -6$$

and the value of ρ for which there is a pole at $\bar{s}$ is given by

$$|\rho_m| = \frac{3 \cdot 4 \cdot 5}{9} = \frac{20}{3}$$

The closed-loop system is thus asymptotically stable for $-\dfrac{20}{3} < \rho < 2$.

Exercise 7

Given the following plant

$$G(s) = \frac{1}{(s+2)(s-2)}$$

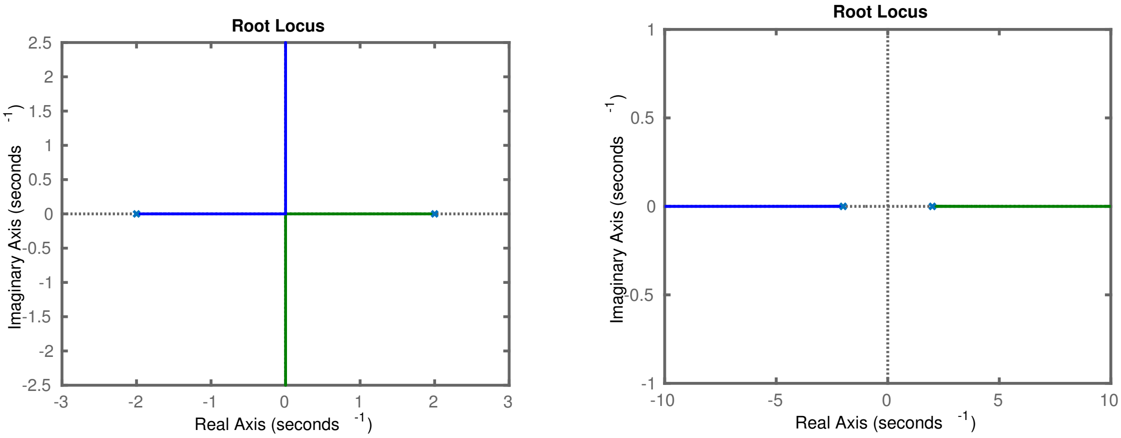
Design a regulator $R(s)$ that stabilizes the closed-loop system, placing two complex poles with natural frequency $\omega_n = \sqrt{2}$ and damping $\xi = \dfrac{1}{\sqrt{2}}$.

Solution

We start considering an algebraic regulator $R(s) = \rho$. The loop transfer function is given by

$$L(s) = \rho \frac{1}{(s+2)(s-2)}$$

The direct and inverse loci are shown in Fig. 7.



(a) direct locus

(b) inverse locus

Figure 7: Direct and inverse root loci.

The asymptotes are characterised by the following angles

$$\theta_a = \begin{cases} \dfrac{180^\circ + h \cdot 360^\circ}{2} = 90^\circ, 270^\circ \\ \dfrac{h \cdot 360^\circ}{2} = 0^\circ, 180^\circ \end{cases}$$

and cross the real axis at

$$x_a = \frac{0 - (2 - 2)}{2} = 0$$

In the direct and inverse locus there is always a pole in the right half plane. The closed-loop system is thus never asymptotically stable.
We have to modify the regulator, including a dynamic part. The direct root locus in Fig. 7 suggests that the asymptote can be moved to the left half plane moving the pole at -2 to the left. Considering a regulator

$$R(s) = \rho \frac{s + 2}{s + p}$$

the pole at -2 is moved to $-p$.
The loop transfer function is given by

$$L(s) = \rho \frac{1}{(s + p)(s - 2)}$$

The asymptote crosses the real axis at

$$x_a = \frac{0 - (p - 2)}{2} = \frac{2 - p}{2}$$

In order to have two complex closed-loop poles characterised by a natural frequency $\omega_n = \sqrt{2}$ and damping $\xi = \dfrac{1}{\sqrt{2}}$, i.e., a real part equal to -1, x_a should be equal to -1 and thus $p = 4$. The direct root locus with

$$R(s) = \rho \frac{s + 2}{s + 4}$$

is shown in Fig. 8

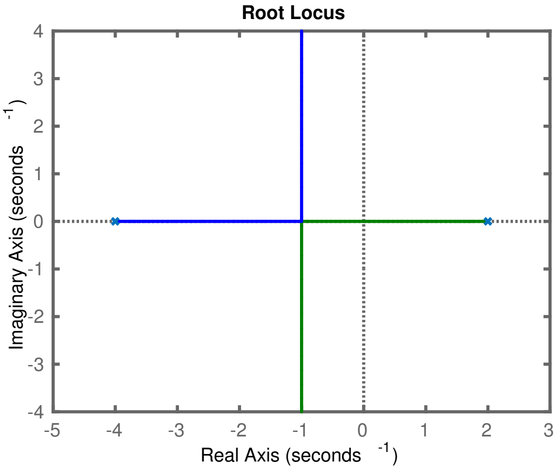


Figure 8: Direct root locus with $R(s) = \rho \dfrac{s+2}{s+4}$.

The two complex poles characterised by a natural frequency $\omega_n = \sqrt{2}$ and damping $\xi = \dfrac{1}{\sqrt{2}}$ have a real part equal to -1 and an imaginary part equal to $\pm j$.
In order to compute the value of ρ that places the poles at $-1 \pm j$ we have to compute the distance from $-1 + j$ to -4 and 2, it follows

$$\rho = \sqrt{3^2 + 1^2} \cdot \sqrt{3^2 + 1^2} = 10$$

The regulator transfer function is thus given by

$$R(s) = 10 \frac{s+2}{s+4}$$

Exercise 8

Given the following linear and time invariant dynamical system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_1(t) - 2x_2(t) + u(t) \\ y(t) &= x_2(t) \end{aligned}$$

Design a pole placement controller that places the closed-loop poles at -2 and -4, and a state estimator whose error dynamics is characterised by two real poles at -20 and -40.
Is the overall controller asymptotically stable?

Solution

The matrices that describe the system in the state space are

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

It is straightforward to verify that (A, B) is completely controllable and (A, C) completely observable.
We start designing the state estimator.
The characteristic polynomial associated to matrix A is

$$\chi_A(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda + 2 \end{vmatrix} = \lambda^2 + 2\lambda - 1$$

The desired characteristic polynomial for the error dynamics is

$$\chi^o(\lambda) = (\lambda + 20)(\lambda + 40) = \lambda^2 + 60\lambda + 800$$

The gains of the state estimator, assuming the system in controllable canonical form, are given by

$$\hat{\ell}_1 = -1 - 800 = -801$$
$$\hat{\ell}_2 = 2 - 60 = -58$$

As the matrices of the state estimator

$$A^T = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \qquad C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are in controllable canonical form, the gain is given by

$$L^T = \begin{bmatrix} -801 & -58 \end{bmatrix}$$

We consider now the pole placement design.
The desired characteristic polynomial for the closed-loop system is

$$\chi^o(\lambda) = (\lambda + 2)(\lambda + 4) = \lambda^2 + 6\lambda + 8$$

The gains of the pole placement law, assuming the system in controllable canonical form, are given by

$$\hat{k}_1 = -1 - 8 = -9$$
$$\hat{k}_2 = 2 - 6 = -4$$

As matrices (A, B) are in controllable canonical form, the gain of the pole placement law is given by

$$K = \begin{bmatrix} -9 & -4 \end{bmatrix}$$

Finally, the state matrix of the controller is

$$A + BK + LC = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -9 & -4 \end{bmatrix} + \begin{bmatrix} -801 \\ -58 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -800 \\ -8 & -64 \end{bmatrix}$$

The characteristic polynomial of the controller is thus

$$\chi_{A+BK+LC} = \begin{vmatrix} \lambda & 800 \\ 8 & \lambda + 64 \end{vmatrix} = \lambda^2 + 64\lambda - 6400$$

and we conclude that the controller is not asymptotically stable.

Exercise 9

Given the following linear and time invariant dynamical system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Design a pole placement controller that increases the damping of the poles so that the closed-loop poles have a damping of $\frac{1}{\sqrt{2}}$, and a state estimator whose error dynamics are 10 times faster than the system dynamics.

Solution

We first need to compute a realization of the transfer function $G(s)$. Using the properties of the Laplace transform we can obtain the following differential equation

$$Y(s) = G(s)U(s) \quad \Rightarrow \quad \ddot{y}(t) + 2\xi\omega_n \dot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t)$$

Defining now $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, we obtain the following state space system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\omega_n^2 x_1(t) - 2\xi\omega_n x_2(t) + \omega_n^2 u(t) \\ y(t) &= x_1(t) \end{aligned}$$

The matrices that describe the system in the state space are

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

It is straightforward to verify that (A, B) is completely controllable and (A, C) completely observable.
We start designing the pole placement law.
The characteristic polynomial associated to matrix A is

$$\chi_A(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ \omega_n^2 & \lambda + 2\xi\omega_n \end{vmatrix} = \lambda^2 + 2\xi\omega_n\lambda + \omega_n^2$$

The desired characteristic polynomial for the closed-loop system is

$$\chi^o(\lambda) = \lambda^2 + \omega_n\sqrt{2}\lambda + \omega_n^2$$

The gains of the pole placement law, assuming the system in controllable canonical form, are given by

$$\hat{k}_1 = \omega_n^2 - \omega_n^2 = 0$$
$$\hat{k}_2 = 2\xi\omega_n - \omega_n\sqrt{2} = \omega_n(2\xi - \sqrt{2})$$

As matrices (A, B) are not in controllable canonical form, we have to compute the transformation matrix that puts the system in canonical form. $\hat{A}$ and $\hat{B}$ are given by

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \qquad \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the controllability matrix

$$\hat{K}_R = \begin{bmatrix} 0 & 1 \\ 1 & -2\xi\omega_n \end{bmatrix}$$

On the other side, the controllability matrix of the original system has the following expression

$$K_R = \begin{bmatrix} 0 & \omega_n^2 \\ \omega_n^2 & -2\xi\omega_n^3 \end{bmatrix}$$

Finally, the transformation matrix is given by

$$T = \hat{K}_R K_R^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} 2\xi\omega_n^3 & \omega_n^2 \\ \omega_n^2 & 0 \end{bmatrix} \frac{1}{\omega_n^4} = \begin{bmatrix} \frac{1}{\omega_n^2} & 0 \\ 0 & \frac{1}{\omega_n^2} \end{bmatrix}$$

The gain of the pole placement law is given by

$$K = \begin{bmatrix} 0 & \omega_n(2\xi - \sqrt{2}) \end{bmatrix} \begin{bmatrix} \frac{1}{\omega_n^2} & 0 \\ 0 & \frac{1}{\omega_n^2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2\xi - \sqrt{2}}{\omega_n} \end{bmatrix}$$

We consider now the state observer design.
The desired characteristic polynomial for the error dynamics is

$$\chi^o(\lambda) = \lambda^2 + 10\omega_n\sqrt{2}\lambda + 100\omega_n^2$$

The gains of the state estimator, assuming the system in controllable canonical form, are given by

$$\hat{\ell}_1 = \omega_n^2 - 100\omega_n^2 = -99\omega_n^2$$
$$\hat{\ell}_2 = 2\xi\omega_n - 10\sqrt{2}\omega_n = (2\xi - 10\sqrt{2})\omega_n$$

As matrices (A^T, C^T)

$$A^T = \begin{bmatrix} 0 & -\omega_n^2 \\ 1 & -2\xi\omega_n \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are not in controllable canonical form, we have to compute the transformation matrix that puts the system in canonical form. $\hat{A}^T$ and $\hat{C}^T$ are given by

$$\hat{A}^T = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \qquad \hat{C}^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the observability matrix

$$K_O = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On the other side, the controllability matrix of the original system has the following expression

$$\hat{K}_O = \begin{bmatrix} 0 & 1 \\ 1 & -2\xi\omega_n \end{bmatrix}$$

Finally, the transformation matrix is given by

$$T = \hat{K}_O K_O^{-1} = \hat{K}_O$$

The gain of the state observer law is given by

$$L = T^T \hat{L} = \begin{bmatrix} 0 & 1 \\ 1 & -2\xi\omega_n \end{bmatrix} \begin{bmatrix} -99\omega_n^2 \\ (2\xi - 10\sqrt{2})\omega_n \end{bmatrix} = \begin{bmatrix} (2\xi - 10\sqrt{2})\omega_n \\ -99\omega_n^2 - 2\xi(2\xi - 10\sqrt{2})\omega_n^2 \end{bmatrix}$$