POLITECNICO DI MILANO





# Automatic Control

## Motion planning

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# Motivations



Electric motors are used in many different applications, ranging from machine tools and industrial robots to household appliances and advanced driver assistance systems.

The motion of these motors should be appropriately controlled, starting from the trajectory performed by the rotor.

A trajectory can be characterized by different profiles and maximum values of velocity, acceleration and jerk, generating different effects on the actuator, on the motion transmission system and on the mechanical load performance and wear.

Motion planning is the activity aiming at selecting suitable velocity, acceleration and jerk profiles.

# Profile selection criteria



Which are the guidelines to select the profile?

- low computational complexity and memory consumption
- continuity of position, velocity (and jerk) profiles
- minimize undesired effects (curvature regularity)
- accuracy (no overshoot)

We will consider two different problems:

- point-to-point motion planning

  only start and goal, and motion duration are specified

- trajectory planning

  a set of desired positions is specified

# Polynomial trajectories (I)



Let's consider the point-to-point motion planning problem.

Given the initial and final conditions on position, velocity, acceleration and jerk, and the motion duration, the easiest solution to the planning problem is given by polynomial functions

$$q(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

The coefficients can be determined imposing the initial and final conditions.

Increasing the order of the polynomial, the trajectory becomes smoother and we can satisfy more initial and final conditions.

We will now consider two classical examples: $3^{rd}$ and $5^{th}$ order polynomials.

# Polynomial trajectories (II)

Given the following initial and final conditions:

- initial and final time ($t_i$ and $t_f$)
- initial position and velocity ($q_i$ and $\dot{q}_i$)
- final position and velocity ($q_f$ and $\dot{q}_f$)

To satisfy the four boundary conditions we need at least a 3rd order polynomial

$$q(t) = a_0 + a_1(t - t_i) + a_2(t - t_i)^2 + a_3(t - t_i)^3$$

Then, imposing the boundary conditions

$$q(t_i) = q_i \quad \dot{q}(t_i) = \dot{q}_i \quad q(t_f) = q_f \quad \dot{q}(t_f) = \dot{q}_f$$

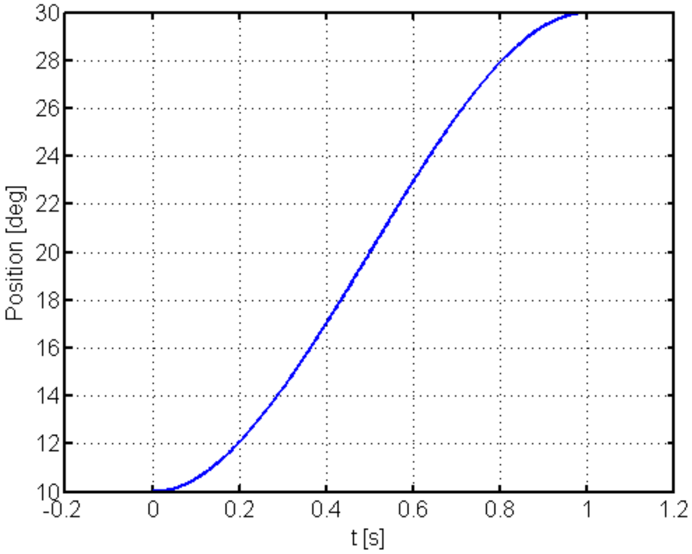
we obtain ($T = t_f - t_i$)

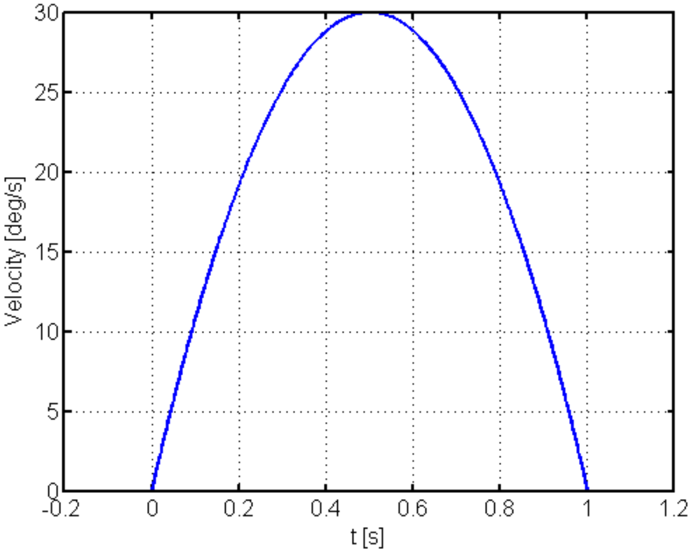
$$a_0 = q_i \quad a_1 = \dot{q}_i \quad a_2 = \frac{-3(q_i - q_f) - (2\dot{q}_i + \dot{q}_f)T}{T^2}$$

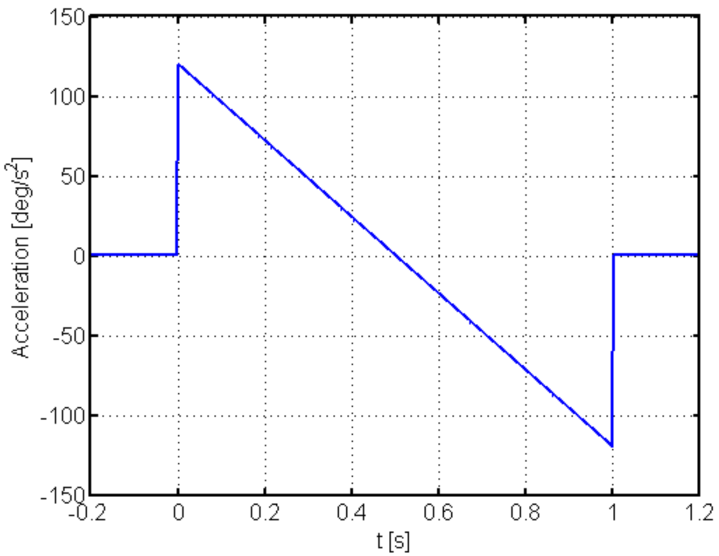
$$a_3 = \frac{2(q_i - q_f) + (\dot{q}_i + \dot{q}_f)T}{T^3}$$

# Polynomial trajectories – Example

$$t_i = 0\,s \quad t_f = 1\,s \quad q_i = 10^\circ \quad q_f = 30^\circ \quad \dot{q}_i = \dot{q}_f = 0^\circ/s$$



In order to enforce initial conditions on the acceleration as well, we need at least a 5th order polynomial

$$q(t) = a_0 + a_1 (t - t_i) + a_2 (t - t_i)^2 + a_3 (t - t_i)^3 + a_4 (t - t_i)^4 + a_5 (t - t_i)^5$$

Then, imposing the boundary conditions

$$q(t_i) = q_i \quad \dot{q}(t_i) = \dot{q}_i \quad \ddot{q}(t_i) = \ddot{q}_i \quad q(t_f) = q_f \quad \dot{q}(t_f) = \dot{q}_f \quad \ddot{q}(t_f) = \ddot{q}_f$$

we obtain ($T = t_f - t_i$)

$$a_0 = q_i \quad a_1 = \dot{q}_i \quad a_2 = \frac{1}{2}\ddot{q}_i$$

$$a_3 = \frac{20\left(q_f - q_i\right) - \left(8\dot{q}_f + 12\dot{q}_i\right)T - \left(3\ddot{q}_f - \ddot{q}_i\right)T^2}{2T^3}$$

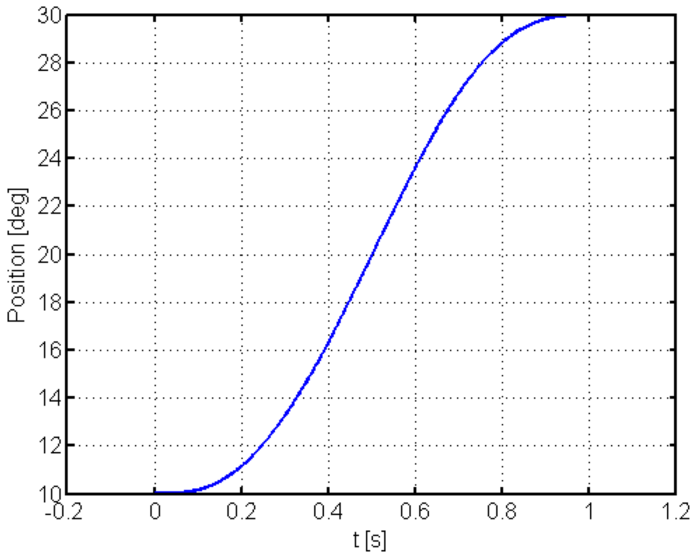
$$a_4 = \frac{30\left(q_i - q_f\right) + \left(14\dot{q}_f + 16\dot{q}_i\right)T + \left(3\ddot{q}_f - 2\ddot{q}_i\right)T^2}{2T^4}$$

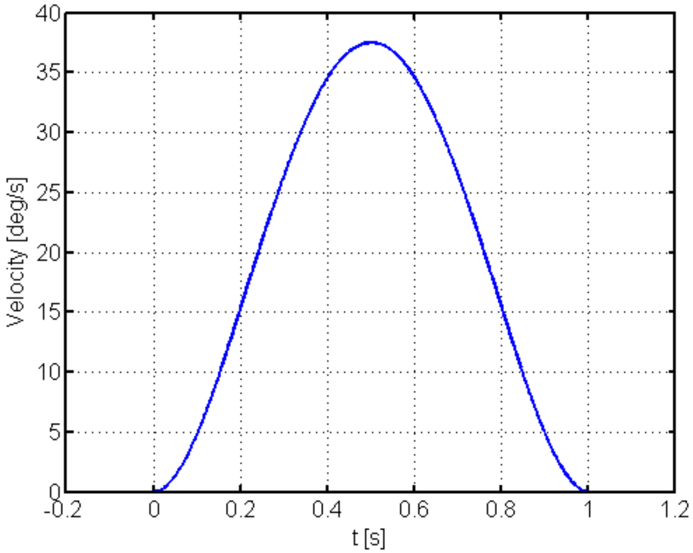
$$a_5 = \frac{12\left(q_f - q_i\right) - 6\left(\dot{q}_f + \dot{q}_i\right)T - \left(\ddot{q}_f - \ddot{q}_i\right)T^2}{2T^5}$$

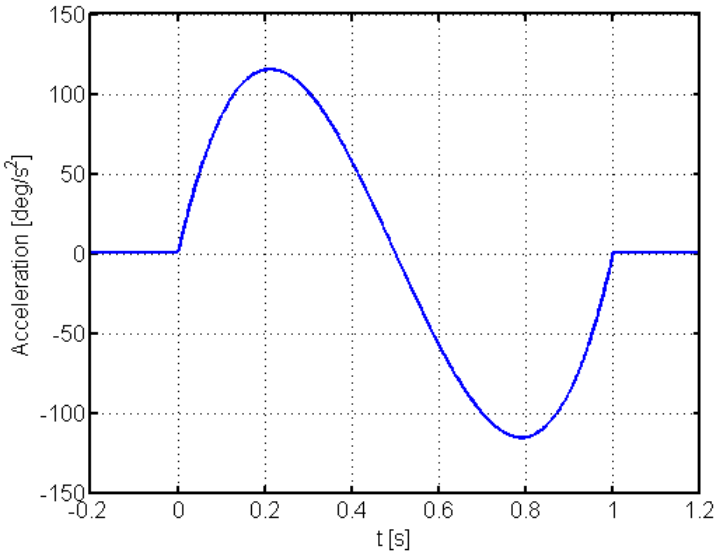
# Polynomial trajectories – Example



$$t_i = 0\,s \quad t_f = 1\,s \quad q_i = 10^\circ \quad q_f = 30^\circ \quad \dot{q}_i = \dot{q}_f = 0^\circ/s \quad \ddot{q}_i = \ddot{q}_f = 0^\circ/s^2$$



# Polynomial trajectories (IV)

An harmonic motion is characterized by an acceleration profile that is proportional to the position profile, with opposite sign.

A generalization of the harmonic trajectory is given by

$$q(t) = \frac{q_f - q_i}{2}\left[1 - \cos\left(\frac{\pi(t - t_i)}{t_f - t_i}\right)\right] + q_i$$

$$\dot{q}(t) = \frac{\pi(q_f - q_i)}{2(t_f - t_i)} \sin\left(\frac{\pi(t - t_i)}{t_f - t_i}\right)$$

$$\ddot{q}(t) = \frac{\pi^2(q_f - q_i)}{2(t_f - t_i)^2} \cos\left(\frac{\pi(t - t_i)}{t_f - t_i}\right)$$

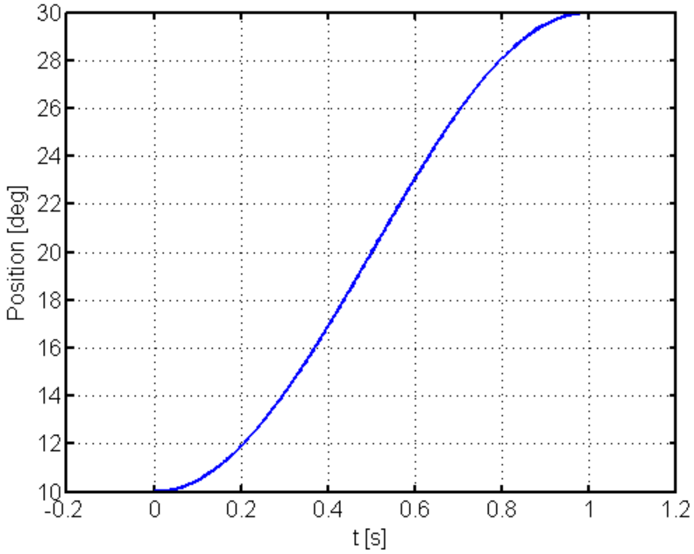
where

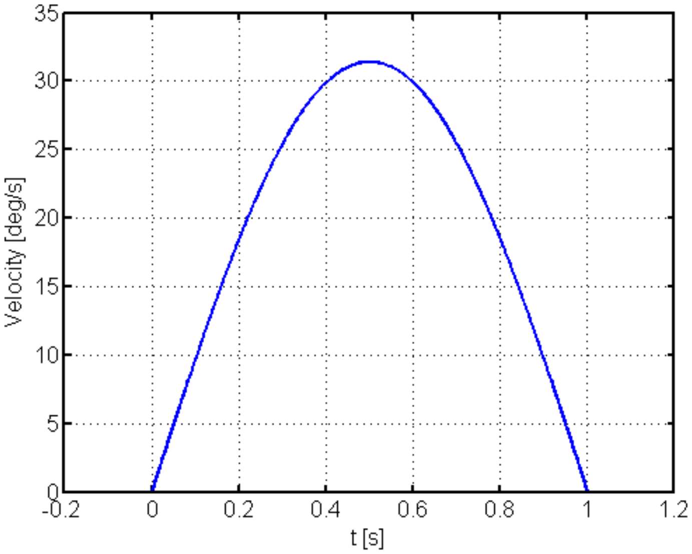
$$q(t_i) = q_i, \quad \dot{q}(t_i) = 0, \quad q(t_f) = q_f, \quad \dot{q}(t_f) = 0$$

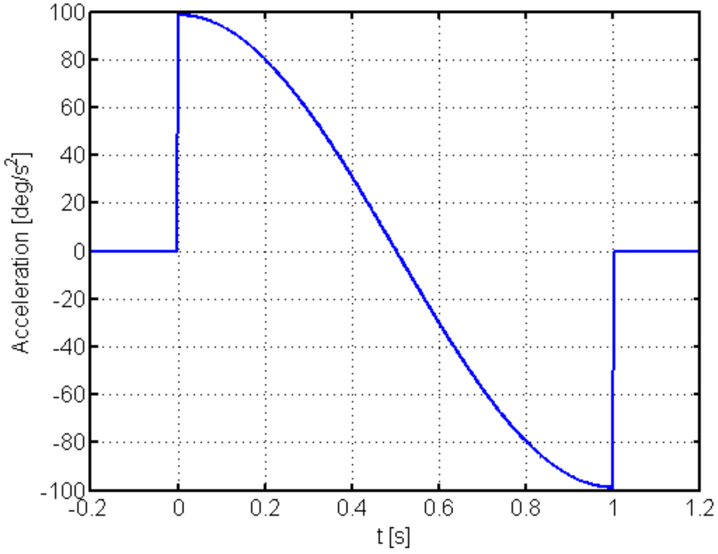
Note that the harmonic trajectory has continuous derivatives (of any order) $\forall t \in (t_i, t_f)$.

# Polynomial trajectories – Example

$$t_i = 0\,s \quad t_f = 1\,s \quad q_i = 10^\circ \quad q_f = 30^\circ$$



# Polynomial trajectories (V)

In order to avoid discontinuities at $t_i$ and $t_f$ in the acceleration profile, we can modify the harmonic trajectory introducing the cycloidal trajectory

$$q(t) = (q_f - q_i)\left[\frac{t-t_i}{t_f-t_i} - \frac{1}{2\pi}\sin\left(\frac{2\pi(t-t_i)}{t_f-t_i}\right)\right] + q_i$$

$$\dot{q}(t) = \frac{q_f - q_i}{t_f - t_i}\left[1 - \cos\left(\frac{2\pi(t-t_i)}{t_f-t_i}\right)\right]$$

$$\ddot{q}(t) = \frac{2\pi(q_f - q_i)}{(t_f-t_i)^2}\sin\left(\frac{2\pi(t-t_i)}{t_f-t_i}\right)$$

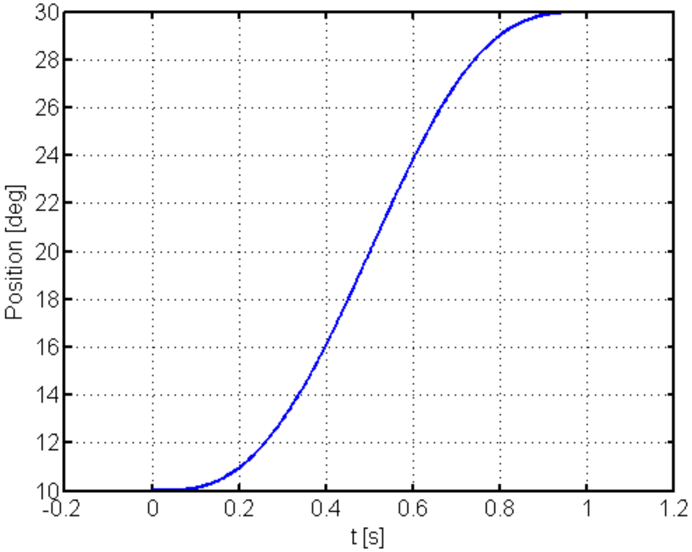
where

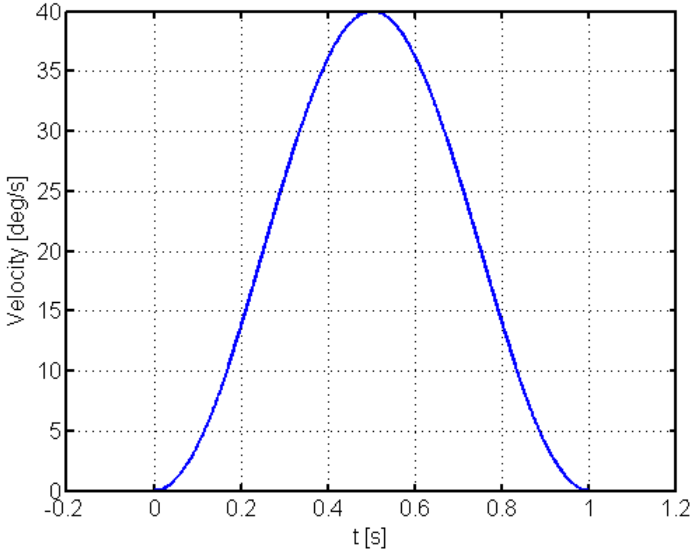
$$q(t_i) = q_i \quad \dot{q}(t_i) = 0 \quad \ddot{q}(t_i) = 0 \quad q(t_f) = q_f \quad \dot{q}(t_f) = 0 \quad \ddot{q}(t_f) = 0$$

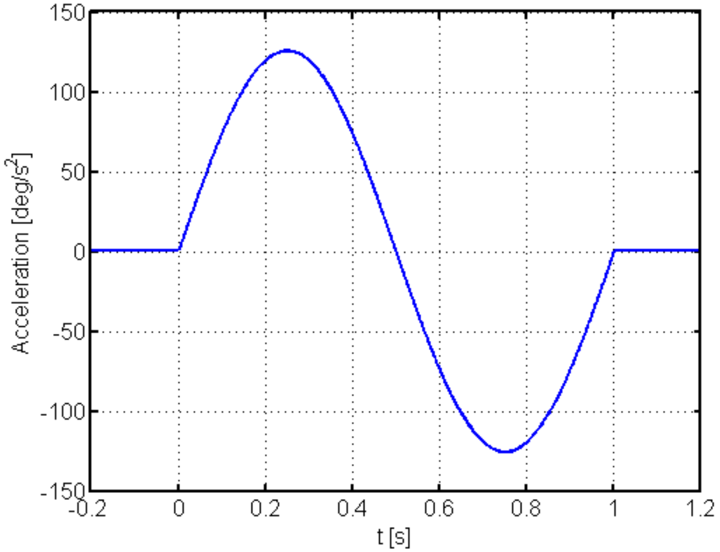
# Polynomial trajectories – Example



$$t_i = 0\,s \quad t_f = 1\,s \quad q_i = 10^\circ \quad q_f = 30^\circ$$



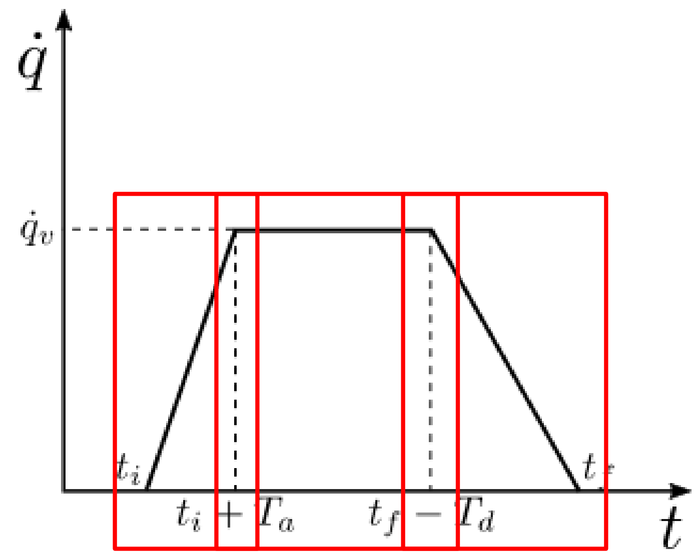
# Trapezoidal trajectories (I)

A very common way to plan the motion of a motor in industrial drives is based on trapezoidal velocity profiles.

The trajectory is composed of a linear part, where the velocity is constant, and two parabolic blends, where the velocity is a linear function of time.

The trajectory can be thus divided in three parts:

1. in the first part a constant acceleration is applied, the velocity is linear and the position a parabolic function of time
2. in the second part the acceleration is zero, the velocity is constant and the position is a linear function of time
3. in the third part a constant negative acceleration is applied, the velocity is linear and the position a parabolic function of time




If $T_a = T_d$ ($T_a \leq (t_f - t_i)/2$) the profile is symmetric.

# Trapezoidal trajectories (II)



First part (acceleration)

$$t \in [t_i, t_i + T_a]$$

$$\ddot{q}(t) = \frac{\dot{q}_v}{T_a}$$

$$\dot{q}(t) = \frac{\dot{q}_v}{T_a}(t - t_i)$$

$$q(t) = q_i + \frac{\dot{q}_v}{2T_a}(t - t_i)^2$$

Second part (constant velocity)

$$t \in [t_i + T_a, t_f - T_d]$$

$$\ddot{q}(t) = 0$$

$$\dot{q}(t) = \dot{q}_v$$

$$q(t) = q_i + \dot{q}_v\left(t - t_i - \frac{T_a}{2}\right)$$

Third part (deceleration)

$$t \in [t_f - T_d, t_f]$$

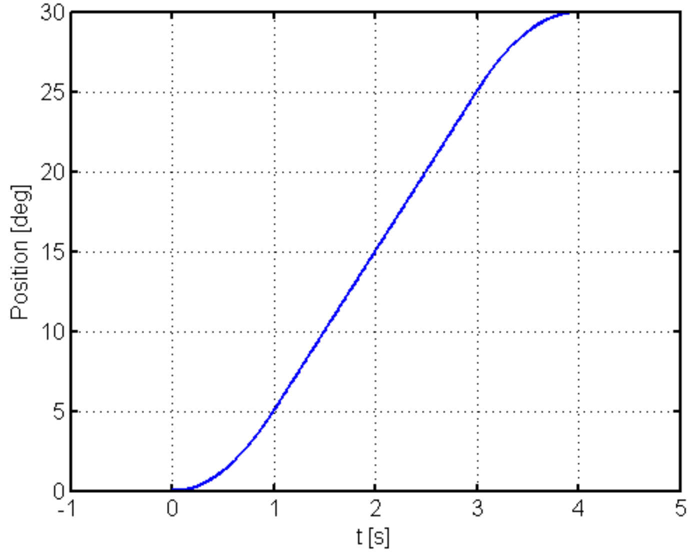
$$\ddot{q}(t) = -\frac{\dot{q}_v}{T_d}$$

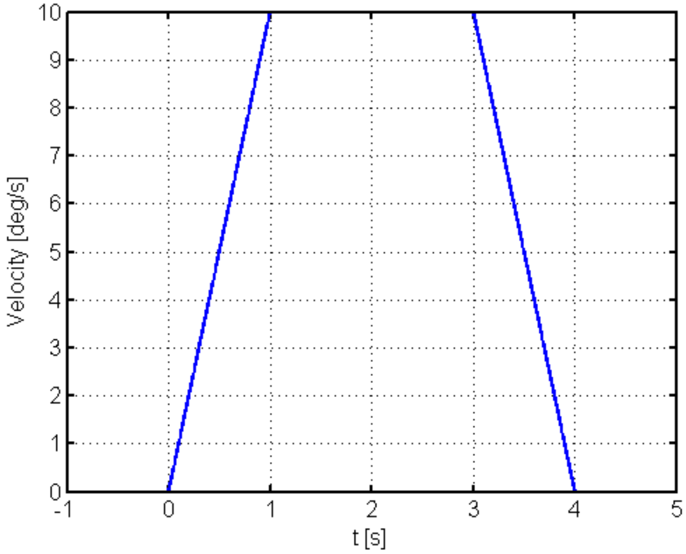
$$\dot{q}(t) = \frac{\dot{q}_v}{T_d}(t_f - t)$$

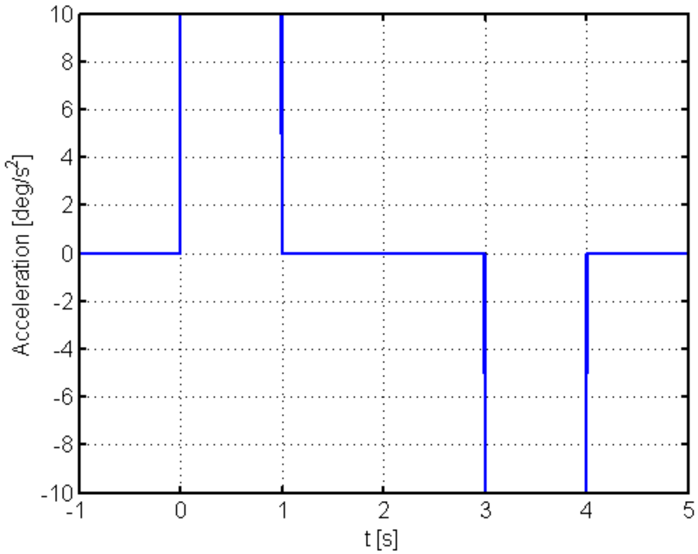
$$q(t) = q_f - \frac{\dot{q}_v}{2T_d}(t_f - t)^2$$

# Trapezoidal trajectories – Example

$$t_i = 0,\ t_f = 4\,s,\ t_a = 1\,s,\ q_i = 0^\circ,\ q_f = 30^\circ,\ \dot{q}_v = 10^\circ/s$$



# Trapezoidal trajectories (III)



In order to correctly select the parameters of the trapezoidal profile, we should satisfy some constraints.

Assuming that $T_d = T_a$, the velocity at time $t_i + T_a$ is given by

$$\ddot{q}_a T_a = \frac{q_m - q_a}{T_m - T_a}$$

where

$$q_a = q\left(t_i + T_a\right) \qquad q_m = \frac{q_i + q_f}{2} \qquad T_m = \frac{t_f - t_i}{2}$$

We also have that

$$q_a = q_i + \frac{1}{2}\ddot{q}_a T_a^2$$

Using the previous relations we obtain the following constraint

$$\ddot{q}_a T_a^2 - \ddot{q}_a\left(t_f - t_i\right)T_a + \left(q_f - q_i\right) = 0$$

Then, integrating the velocity profile we obtain a constraint on the velocities

$$q_f - q_i = \dot{q}_v\left(t_f - t_i - T_a\right)$$

# Trapezoidal trajectories (IV)



Given

- the distance $h = q_f - q_i$
- the duration $T = t_f - t_i$

We have three different ways to specify the trajectory:

1. choosing the acceleration time $T_a$ ($T_a < T/2$)

$$\dot{q}_v = \frac{h}{T - T_a} \qquad \ddot{q}_a = \frac{\dot{q}_v}{T_a}$$

2. choosing the acceleration ($|\ddot{q}_a| \geq 4|h|/T^2$)

$$T_a = \frac{\ddot{q}_a T - \sqrt{\ddot{q}_a^2 T^2 - 4\ddot{q}_a h}}{2\ddot{q}_a} \qquad \dot{q}_v = \ddot{q}_a T_a$$

3. choosing the velocity ($|\dot{q}_v| \geq |h|/T$)

$$T_a = \frac{\dot{q}_v T - h}{\dot{q}_v} \qquad \ddot{q}_a = \frac{\dot{q}_v}{T_a}$$

If we would like to select the maximum velocity and acceleration according to the motor specifications, we have to select

- acceleration time $T_a = \frac{\dot{q}_{max}}{\ddot{q}_{max}}$
- distance $h = \dot{q}_{max}(T - T_a)$

In this case the duration will be

$$T = \frac{h}{\dot{q}_{max}} + \frac{\dot{q}_{max}}{\ddot{q}_{max}}$$

and the trajectory is described by the following equations

$$q(t) = \begin{cases} q_i + \frac{1}{2}\ddot{q}_{max}\left(t - t_i\right)^2 & t_i \le t < t_i + T_a \\ q_i + \ddot{q}_{max}T_a\left(t - t_i - \frac{T_a}{2}\right) & t_i + T_a \le t < t_f - T_a \\ q_f - \frac{1}{2}\ddot{q}_{max}\left(t_f - t\right)^2 & t_f - T_a \le t \le t_f \end{cases}$$

These equations hold only if $T_a \le T/2$, or equivalently $h \ge \dot{q}_{max}^2/\ddot{q}_{max}$.

# Trapezoidal trajectories (VI)

If $h < \dot{q}_{max}^2/\ddot{q}_{max}$ the trajectory will not reach the maximum velocity and the maximum acceleration.

If we would like to minimize the trajectory duration we can use a bang-bang acceleration profile

$$q(t) = \begin{cases} q_i + \frac{1}{2}\ddot{q}_{max}(t - t_i)^2 & t_i \le t < t_i + T_a \\ q_f - \frac{1}{2}\ddot{q}_{max}(t_f - t)^2 & t_f - T_a \le t \le t_f \end{cases}$$

where the acceleration time and the trajectory duration are

$$T_a = \sqrt{\frac{h}{\ddot{q}_{max}}} \qquad T = 2T_a = 2\sqrt{\frac{h}{\ddot{q}_{max}}}$$

and the maximum velocity is

$$\dot{q}_v = \ddot{q}_{max} T_a = \frac{h}{T_a} = 2\frac{h}{T}$$

# Trapezoidal trajectories (VII)

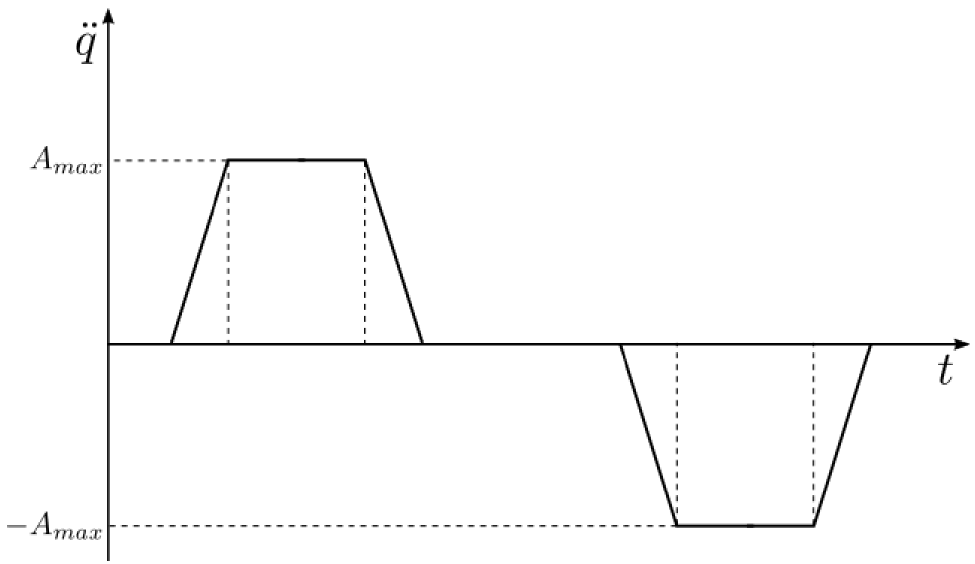
The trapezoidal velocity profile is characterized by a discontinuous acceleration profile. As a consequence, jerk has infinite values in the acceleration discontinuities.

This can negatively affect a mechanical system, increasing wear and causing vibrations.

To overcome this problem we can introduce a continuous trapezoidal acceleration profile.

The trajectory can be divided in three parts:

1. acceleration (acceleration linearly increases until the maximum value and then linearly decreases)
2. constant velocity
3. deceleration (is symmetric to phase 1)



# Trajectory scaling (I)

Sometimes a planned trajectory need to be scaled in order to adapt to actuator constraints.

There are two different solutions to this problem:

1. kinematic scaling, trajectory has to satisfy maximum acceleration and maximum velocity constraints
2. dynamic scaling, trajectory has to satisfy maximum torque constraints

We will now consider the problem of kinematic scaling.

In order to scale a trajectory we need to parametrize it introducing a normalized parameter $\sigma = \sigma(t)$.

Given a trajectory $q(t)$, from $q_i$ to $q_f$, whose duration is $T = t_f - t_i$, the normalized form is

$$q(t) = q_i + h\sigma(\tau)$$

where $h = q_f - q_i$ and

$$0 \leq \sigma(\tau) \leq 1 \qquad \tau = \frac{t - t_i}{T} \quad 0 \leq \tau \leq 1$$

# Trajectory scaling (II)



Considering the normalized form

$$q(t) = q_i + h\sigma(\tau)$$

we have

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \frac{h}{T}\sigma'(\tau)$$

$$\frac{\mathrm{d}^2 q(t)}{\mathrm{d}t^2} = \frac{h}{T^2}\sigma''(\tau)$$

$$\vdots$$

$$\frac{\mathrm{d}^n q(t)}{\mathrm{d}t^n} = \frac{h}{T^n}\sigma^{(n)}(\tau)$$

The maximum velocity, acceleration, … values correspond to the maximum values of functions $\sigma^{(i)}(\tau)$.

Modifying the trajectory duration $T$ one can satisfy the kinematic constraints.

# Trajectory scaling (III)



Consider a 3rd order polynomial trajectory, we can introduce the parameter

$$\sigma(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3$$

Imposing the boundary conditions

$$\sigma(0) = \sigma'(0) = \sigma'(1) = 0 \qquad \sigma(1) = 1$$

we obtain

$$a_0 = a_1 = 0 \qquad a_2 = 3 \qquad a_3 = -2$$

and consequently

$$\sigma(\tau) = 3\tau^2 - 2\tau^3 \qquad \sigma'(\tau) = 6\tau - 6\tau^2$$

$$\sigma''(\tau) = 6 - 12\tau \qquad \sigma'''(\tau) = -12$$

Maximum velocity and acceleration are given by

$$\sigma'_{max} = \sigma'(0.5) = \frac{3}{2} \quad \Rightarrow \quad \dot{q}_{max} = \frac{3h}{2T}$$

$$\sigma''_{max} = \sigma''(0) = 6 \quad \Rightarrow \quad \ddot{q}_{max} = \frac{6h}{T^2}$$

# Trajectory scaling (IV)

Consider a 5th order polynomial trajectory, we can introduce the parameter

$$\sigma(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5$$

Imposing the boundary conditions

$$\sigma(0) = \sigma'(0) = \sigma''(0) = 0 \qquad \sigma(1) = 1 \qquad \sigma'(1) = \sigma''(1) = 0$$

we obtain

$$a_0 = a_1 = a_2 = 0 \qquad a_3 = 10 \qquad a_4 = -15 \qquad a_5 = 6$$

and consequently

$$\sigma(\tau) = 10\tau^3 - 15\tau^4 + 6\tau^5 \qquad \sigma'(\tau) = 30\tau^2 - 60\tau^3 + 30\tau^4$$

$$\sigma''(\tau) = 60\tau - 180\tau^2 - 120\tau^3 \qquad \sigma'''(\tau) = 60 - 360\tau + 360\tau^2$$

Maximum velocity, acceleration and jerk are given by

$$\sigma'_{max} = \sigma'(0.5) = \frac{15}{8} \quad \Rightarrow \quad \dot{q}_{max} = \frac{15h}{8T}$$

$$\sigma''_{max} = \sigma''(0.2123) = \frac{10\sqrt{3}}{3} \quad \Rightarrow \quad \ddot{q}_{max} = \frac{10\sqrt{3}h}{3T^2}$$

$$\sigma'''_{max} = \sigma'''(0) = 60 \quad \Rightarrow \quad \dddot{q}_{max} = \frac{60h}{T^3}$$

# Trajectory scaling (V)

A harmonic trajectory can be parameterized introducing

$$\sigma(\tau) = \frac{1}{2}\left(1 - \cos(\pi\tau)\right)$$

consequently

$$\sigma'(\tau) = \frac{\pi}{2}\sin(\pi\tau) \qquad \sigma''(\tau) = \frac{\pi^2}{2}\cos(\pi\tau) \qquad \sigma'''(\tau) = \frac{\pi^3}{2}\sin(\pi\tau)$$

Maximum velocity, acceleration and jerk are given by

$$\sigma'_{max} = \sigma'(0.5) = \frac{\pi}{2} \quad \Rightarrow \quad \dot{q}_{max} = \frac{\pi h}{2T}$$

$$\sigma''_{max} = \sigma''(0) = \frac{\pi^2}{2} \quad \Rightarrow \quad \ddot{q}_{max} = \frac{\pi^2 h}{2T^2}$$

$$\sigma'''_{max} = \sigma'''(0.5) = \frac{\pi^3}{2} \quad \Rightarrow \quad \dddot{q}_{max} = \frac{\pi^3 h}{2T^3}$$

# Trajectory scaling (VI)



A cycloidal trajectory can be parameterized introducing

$$\sigma(\tau) = \tau - \frac{1}{2\pi}\sin(2\pi\tau)$$

consequently

$$\sigma'(\tau) = 1 - \cos(2\pi\tau)$$

$$\sigma''(\tau) = 2\pi\sin(2\pi\tau)$$

$$\sigma'''(\tau) = 4\pi^2\cos(2\pi\tau)$$

Maximum velocity, acceleration and jerk are given by

$$\sigma'_{max} = \sigma'(0.5) = 2 \quad \Rightarrow \quad \dot{q}_{max} = \frac{2h}{T}$$

$$\sigma''_{max} = \sigma''(0.25) = 2\pi \quad \Rightarrow \quad \ddot{q}_{max} = \frac{2\pi h}{T^2}$$

$$\sigma'''_{max} = \sigma'''(0) = 4\pi^2 \quad \Rightarrow \quad \dddot{q}_{max} = \frac{4\pi^2 h}{T^3}$$

# Trajectory scaling – Example

Consider a trajectory with $q_i = 10°$, $q_f = 50°$, and a motor characterized by $\dot{q}_{max} = 30\,°/s$ and $\ddot{q}_{max} = 80\,°/s^2$.

| Trajectory | Max vel/acc | Constraints | Min duration |
|---|---|---|---|
| 3rd deg. poly | $\dot{q}_{max} = \frac{3h}{2T}$<br>$\ddot{q}_{max} = \frac{6h}{T^2}$ | $T \geq \frac{3h}{60} = 2$<br>$T \geq \sqrt{\frac{6h}{80}} = 1.732$ | 2 |
| 5th deg. poly | $\dot{q}_{max} = \frac{15h}{8T}$<br>$\ddot{q}_{max} = \frac{10\sqrt{3}h}{3T^2}$ | $T \geq \frac{15h}{240} = 2.5$<br>$T \geq \sqrt{\frac{10\sqrt{3}h}{240}} = 1.699$ | 2.5 |
| Harmonic | $\dot{q}_{max} = \frac{\pi h}{2T}$<br>$\ddot{q}_{max} = \frac{\pi^2 h}{2T^2}$ | $T \geq \frac{\pi h}{60} = 2.094$<br>$T \geq \sqrt{\frac{\pi^2 h}{160}} = 1.571$ | 2.094 |
| Cycloidal | $\dot{q}_{max} = \frac{2h}{T}$<br>$\ddot{q}_{max} = \frac{2\pi h}{T^2}$ | $T \geq \frac{2h}{30} = 2.667$<br>$T \geq \sqrt{\frac{2\pi h}{80}} = 1.772$ | 2.667 |

Faster (3rd deg. poly)

Faster (Harmonic)

# Interpolation (I)

Up to now we have considered only point-to-point trajectories. In many practical problems, however, we would like to determine a trajectory that goes through multiple points.

This problem can be addressed using interpolation techniques.

We are now interested to determine a trajectory that goes through a set of points at certain time instants

$$\begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n-1} \\ t_n \end{bmatrix} \Rightarrow \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{n-1} \\ q_n \end{bmatrix}$$

# Interpolation (II)

To determine a trajectory that goes through $n$ points we can consider a polynomial function of order $n-1$

$$q(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}$$

Given a set of points $t_i, q_i$ $i = 1, \dots, n$, we can define vectors $\mathbf{q}$ and $\mathbf{a}$, and Vandermonde matrix $\mathbf{T}$ as follows

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{n-1} \\ q_n \end{bmatrix} = \begin{bmatrix} 1 & t_1 & \dots & t_1^{n-1} \\ 1 & t_2 & \dots & t_2^{n-1} \\ & & \vdots & \\ 1 & t_{n-1} & \dots & t_{n-1}^{n-1} \\ 1 & t_n & \dots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{bmatrix}$$

as a consequence

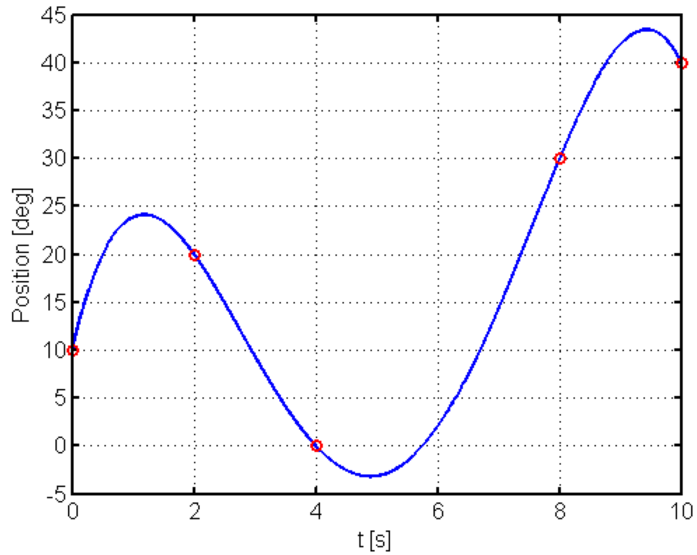
$$\mathbf{a} = \mathbf{T}^{-1} \mathbf{q}$$

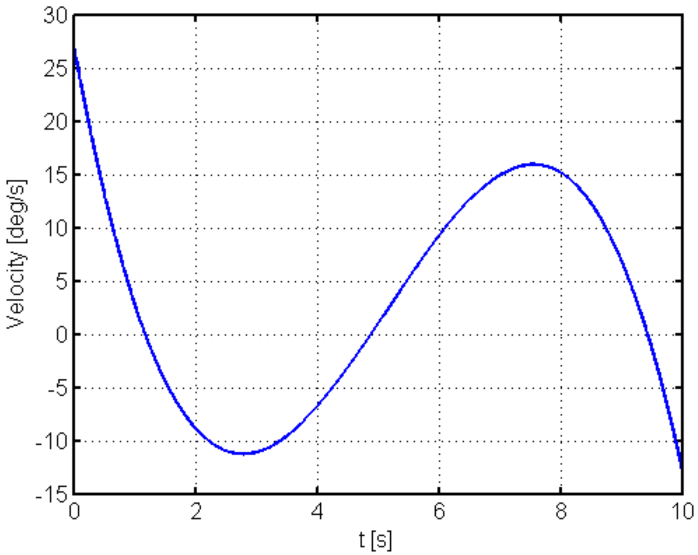
*Caveat*: matrix $T$ is always invertible if $t_i > t_{i-1}$ $i = 1, \dots, n$.

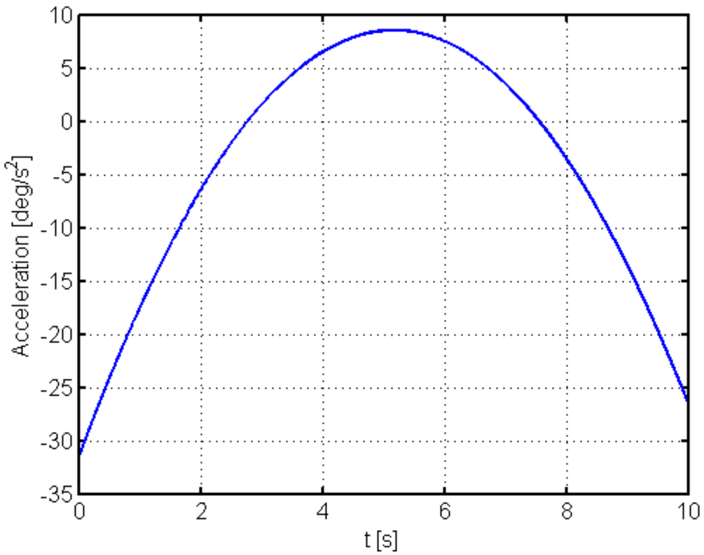
# Interpolation – Example

Polynomial interpolation with 4th order polynomial

| $t_i$ | 0 | 2 | 4 | 8 | 10 |
|---|---|---|---|---|---|
| $q_i$ | 10° | 20° | 0° | 30° | 40° |



# Interpolation (III)



Pro & cons of the polynomial interpolation method:

- $q(t)$ derivatives of any order are continuous in $(t_1, t_n)$
- increasing the number of points the condition number of matrix $\mathbf{T}$ increases, making the computation of the inverse an ill-conditioned problem

For example, assuming $t_i = i/n, i = 1, \dots, n$

| $n$ | 3 | 4 | 5 | 6 | 10 | 15 | 20 |
|---|---|---|---|---|---|---|---|
| Condition number | $15.1$ | $98.87$ | $686.43$ | $4924.37$ | $1.519 \cdot 10^7$ | $4.032 \cdot 10^{11}$ | $1.139 \cdot 10^{16}$ |

Other methods exist to compute the coefficients of $q(t)$, but they always become more and more inefficient as $n$ increases.

# Interpolation (IV)



Other issues of polynomial interpolation are:

- the computational complexity of determining the coefficients increases as $n$ increases
- if a single point $t_i, q_i$ changes we need to recompute all the coefficients
- if we add a point at the end of the trajectory $t_{n+1}, q_{n+1}$ we need to increase the order of the polynomial and recompute all the coefficients
- the trajectory obtained with polynomial interpolation is usually affected by undesired oscillations

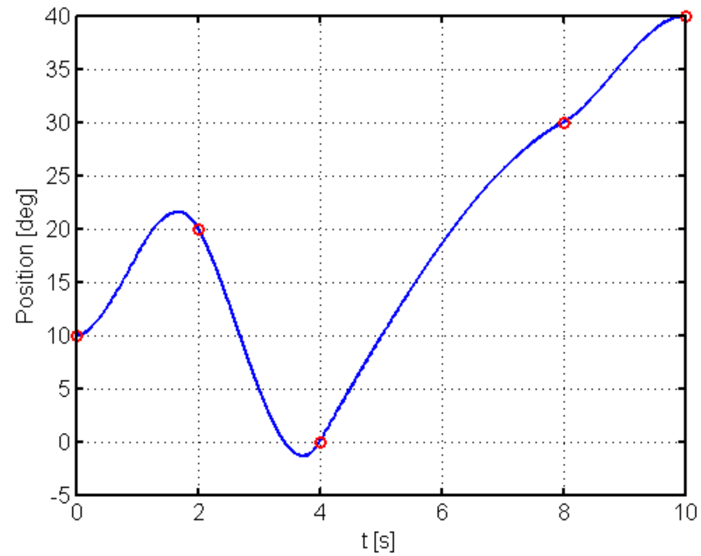
In order to overcome these issues we can consider $n-1$ polynomials of order $p$ (instead of just one polynomial), each one interpolating a segment of the trajectory.

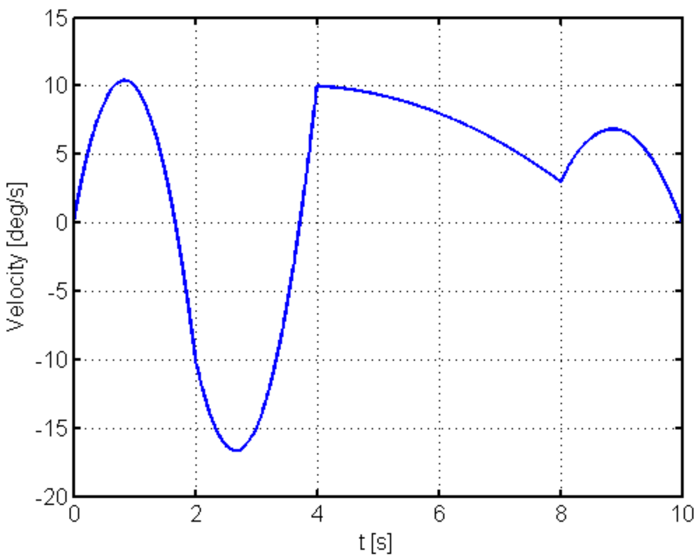
A first solution can be obtained using 3$^{rd}$ order polynomials and imposing position and velocity constraints for each point of the trajectory, in order to compute the cubic polynomial between two consecutive points.

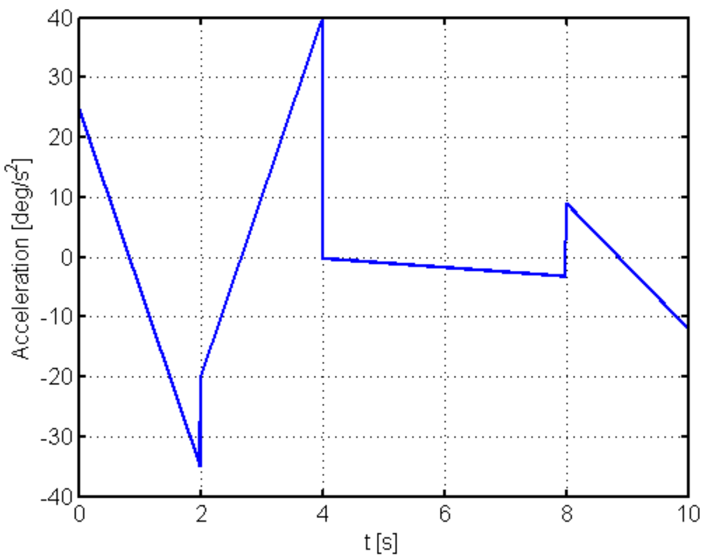
# Interpolation – Example

Polynomial interpolation with 3rd order polynomial segments

| $t_i$ | 0 | 2 | 4 | 8 | 10 |
|---|---|---|---|---|---|
| $q_i$ | 10° | 20° | 0° | 30° | 40° |
| $\dot{q}_i$ | 0°/s | -10°/s | 10°/s | 3°/s | 0°/s |



# Interpolation (V)

If only the desired points are specified, we can compute the velocities using the following relations

$$\dot{q}_1 = 0$$

$$\dot{q}_k = \begin{cases} 0 & \text{sign}(R_k) \neq \text{sign}(R_{k+1}) \\ \frac{R_k + R_{k+1}}{2} & \text{sign}(R_k) = \text{sign}(R_{k+1}) \end{cases}$$

$$\dot{q}_n = 0$$

where

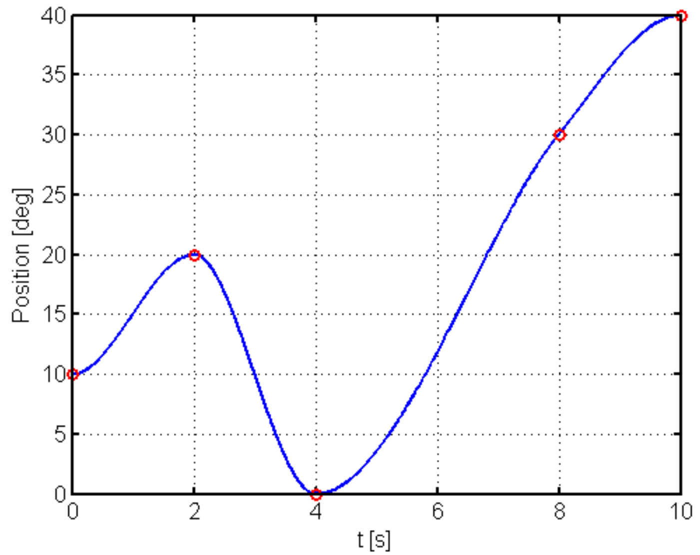
$$R_k = \frac{q_k - q_{k-1}}{t_k - t_{k-1}}$$

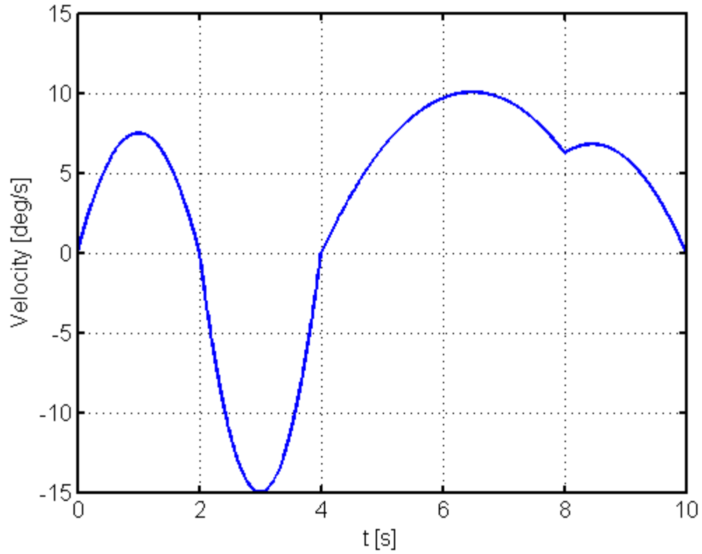
is the slope between points at $t_{k-1}$ and $t_k$.

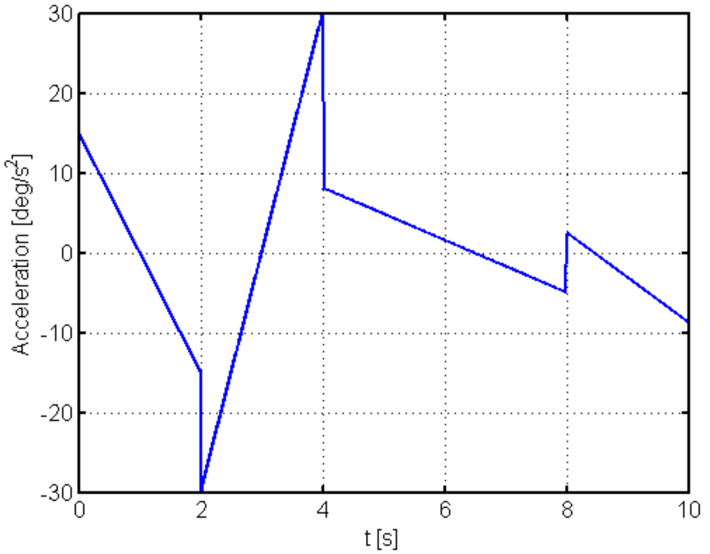
# Interpolation – Example

Polynomial interpolation with 3rd order polynomial segments

| $t_i$ | 0 | 2 | 4 | 8 | 10 |
|---|---|---|---|---|---|
| $q_i$ | 10° | 20° | 0° | 30° | 40° |



# Interpolation (VI)



As you have seen in the example, a trajectory generated with cubic interpolation is affected by discontinuities in the acceleration.

To overcome this problem we can still use cubic segments, but we should specify only the position of each point and impose the continuity of position, velocity and acceleration.

This approach leads to the so called <u>smooth path line</u> (spline).

Among all the interpolating functions ensuring continuity of the derivatives, the spline is the one that has minimum curvature.

Let's study now how to determine the coefficients of the cubic segments that form the spline.

# Interpolation (VII)

If we consider $n$ points we will have $n-1$ cubic polynomials

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

each one characterized by 4 coefficients. We have thus to determine $4(n-1)$ coefficients, imposing the following constraints:

- $2(n-1)$ interpolation conditions (every cubic segment should start and end in a trajectory point)
- $n-2$ continuity conditions on the velocity of the inner points
- $n-2$ continuity conditions on the accelerations of the inner points

At the end we have $4(n-1)$ coefficients and only $4n-6$ constraints.

We can enforce two more constraints imposing, for example, the initial and final velocity.

Let's summarize the problem.

# Interpolation (VIII)

We would like to determine a function

$$q(t) = \{q_k(t), \quad t \in [t_k, t_{k+1}], \quad k = 1, \ldots, n-1\}$$

where each segment is described by

$$q_k(\tau) = a_{k0} + a_{k1}\tau + a_{k2}\tau^2 + a_{k3}\tau^3 \quad \tau \in [0, T_k]$$

given the following constraints

$$q_k(0) = q_k \quad q_k(T_k) = q_{k+1} \qquad k = 1, \ldots, n-1$$

$$\dot{q}_k(T_k) = \dot{q}_{k+1}(0) = v_{k+1} \qquad k = 1, \ldots, n-2$$

$$\ddot{q}_k(T_k) = \ddot{q}_{k+1}(0) \qquad k = 1, \ldots, n-2$$

$$\dot{q}_1(0) = v_1 \quad \dot{q}_{n-1}(T_{n-1}) = v_n$$

where $v_k, k = 2, \ldots, n-1$ must be specified.

To solve this problem we have to determine coefficients $a_{ki}$.

We start assuming velocities $v_k, k = 2, \dots, n-1$ known.

For each cubic polynomial we have 4 boundary conditions on positions and velocities, giving rise to the following system

$$q_k(0) = a_{k0} = q_k$$

$$\dot{q}_k(0) = a_{k1} = v_k$$

$$q_k(T_k) = a_{k0} + a_{k1}T_k + a_{k2}T_k^2 + a_{k3}T_k^3 = q_{k+1}$$

$$\dot{q}_k(T_k) = a_{k1} + 2a_{k2}T_k + 3a_{k3}T_k^2 = v_{k+1}$$

Solving this system with respect to the coefficients yields

$$a_{k0} = q_k$$

$$a_{k1} = v_k$$

$$a_{k2} = \frac{1}{T_k}\left[\frac{3\left(q_{k+1} - q_k\right)}{T_k} - 2v_k - v_{k+1}\right]$$

$$a_{k3} = \frac{1}{T_k^2}\left[\frac{2\left(q_k - q_{k+1}\right)}{T_k} + v_k + v_{k+1}\right]$$

In order to compute velocities $v_k, k = 2, \dots, n-1$ we impose $n$ continuity conditions on the accelerations of the intermediate points

$$\ddot{q}_k(T_k) = 2a_{k2} + 6a_{k3}T_k = 2a_{k+1,2} = \ddot{q}_{k+1}(0) \quad k = 1, \dots, n-2$$

If we now substitute the expressions of coefficients $a_{k2}$, $a_{k3}$, $a_{k+1,2}$, and multiply by $T_k T_{k+1}/2$, we obtain

$$T_{k+1}v_k + 2(T_{k+1} + T_k)v_{k+1} + T_k v_{k+2}$$
$$= \frac{3}{T_k T_{k+1}} \left[T_k^2 \left(q_{k+2} - q_{k+1}\right) + T_{k+1}^2 \left(q_{k+1} - q_k\right)\right]$$

and in matrix form

$$\begin{bmatrix} T_2 & 2(T_1+T_2) & T_1 & & & & \\ 0 & T_3 & 2(T_2+T_3) & T_2 & & & \\ & & & \vdots & & & \\ & & & & T_{n-2} & 2(T_{n-3}+T_{n-2}) & T_{n-3} & 0 \\ & & & & & T_{n-1} & 2(T_{n-2}+T_{n-1}) & T_{n-2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-3} \\ c_{n-2} \end{bmatrix}$$

where $c_k$ are functions of the intermediate positions and the duration of each segment.

Finally, neglecting $v_1$ and $v_n$ that are known, we have

$$\begin{bmatrix} 2(T_1+T_2) & T_1 & & & \\ T_3 & 2(T_2+T_3) & T_2 & & \\ & & \vdots & & \\ & & T_{n-2} & 2(T_{n-3}+T_{n-2}) & T_{n-3} \\ & & & T_{n-1} & 2(T_{n-2}+T_{n-1}) \end{bmatrix} \begin{bmatrix} v_2 \\ \vdots \\ v_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{T_1 T_2}\left[T_1^2(q_3-q_2)+T_2^2(q_2-q_1)\right]-T_2 v_1 \\ \frac{3}{T_2 T_3}\left[T_2^2(q_4-q_3)+T_3^2(q_3-q_2)\right] \\ \vdots \\ \frac{3}{T_{n-3}T_{n-2}}\left[T_{n-3}^2(q_{n-1}-q_{n-2})+T_{n-2}^2(q_{n-2}-q_{n-3})\right] \\ \frac{3}{T_{n-2}T_{n-1}}\left[T_{n-2}^2(q_n-q_{n-1})+T_{n-1}^2(q_{n-1}-q_{n-2})\right]-T_{n-2}v_n \end{bmatrix}$$

This is a set of $n-2$ linear equations that can be written as $\mathbf{A}\mathbf{v}=\mathbf{c}$.

We observe that

- matrix $\mathbf{A}$ is a diagonally dominant matrix, and is always invertible for $T_k > 0$
- matrix $\mathbf{A}$ is a tridiagonal matrix, implying that numerically efficient techniques exist (Gauss-Jordan method) to compute its inverse
- once the inverse of matrix $\mathbf{A}$ is known, we can compute velocities $v_2, \dots, v_{n-1}$ as follows

$$\mathbf{v} = \mathbf{A}^{-1}\mathbf{c}$$

We can derive a similar procedure to determine the spline using the intermediate accelerations instead of the intermediate velocities.

# Interpolation (XIII)



We conclude observing that the total duration of the trajectory is given by

$$T = \sum_{k=1}^{n-1} T_k = t_n - t_1$$

In many applications one is interested to find the minimum time trajectory.

This goal can be achieved determining the segment durations $T_k$ that minimize $T$, satisfying the maximum velocity and maximum acceleration constraints:

$$\min_{T_k} T = \sum_{k=1}^{n-1} T_k$$

$$\text{s.t.} \qquad \left|\dot{q}(\tau, T_k)\right| < v_{max} \quad \tau \in [0, T]$$

$$\left|\ddot{q}(\tau, T_k)\right| < a_{max} \quad \tau \in [0, T]$$

This is a nonlinear optimization problem with linear cost function.

# Interpolation – Example

Cubic spline interpolation

| $t_i$ | 0 | 2 | 4 | 8 | 10 |
|---|---|---|---|---|---|
| $q_i$ | 10° | 20° | 0° | 30° | 40° |