Control of Mobile Robots
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EXERCISE 1

1. Consider a wheel rolling without slipping on the horizontal plane, keeping the sagittal plane in the vertical direction. Write the expression of the pure rolling constraint in the case of a fixed and a steerable wheel.

   The pure rolling constraint has always the same form, independently of the fact that the wheel is fixed or steerable, and is given by
   $$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$
   where x, y are the positions of the wheel contact point with respect to a fixed reference frame, and θ is the wheel sagittal plane orientation.

   Considering two wheels, attached to the same vehicle having orientation θ, the fixed wheel is characterised by a pure rolling constraint
   $$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$
   while the steerable wheel is characterised by a pure rolling constraint
   $$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) = 0$$
   where ϕ is the steering angle.

2. Are the previous constraints holonomic or nonholonomic? Motivate the answer with a mathematical proof.

   The pure rolling constraint
   $$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$
   is a nonholonomic constraint.

   Rewriting the constraint in Pfaffian form
   $$\begin{bmatrix}\sin\theta & -\cos\theta & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = 0$$
   we can show it is nonholonomic using the necessary and sufficient condition. If the constraint were holonomic, we should find a function $\alpha(\mathbf{q})$ that satisfies the following equations
   $$\frac{\partial(\alpha(\mathbf{q})\sin\theta)}{\partial y} = -\frac{\partial(\alpha(\mathbf{q})\cos\theta)}{\partial x} \tag{1}$$
   $$\frac{\partial(\alpha(\mathbf{q})\sin\theta)}{\partial \theta} = 0 \tag{2}$$
   $$0 = -\frac{\partial(\alpha(\mathbf{q})\cos\theta)}{\partial \theta} \tag{3}$$
   From equation (1) we get
   $$\sin\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial y} = -\cos\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$
   from (2)
   $$\sin\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial \theta} + \alpha(\mathbf{q})\cos\theta = 0 \qquad \Rightarrow \qquad -\cos\theta = \frac{\sin\theta}{\alpha(\mathbf{q})}\frac{\partial(\alpha(\mathbf{q}))}{\partial \theta}$$

and from (3)

$$-\cos\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}+\alpha(\mathbf{q})\sin\theta=0 \qquad \Rightarrow \qquad \sin\theta=\frac{\cos\theta}{\alpha(\mathbf{q})}\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}$$

Substituting the previous two expressions in equation (1) we get

$$\cancel{\frac{\cos\theta}{\alpha(\mathbf{q})}\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}}\frac{\partial(\alpha(\mathbf{q}))}{\partial y}=\cancel{\frac{\sin\theta}{\alpha(\mathbf{q})}\frac{\partial(\alpha(\mathbf{q}))}{\partial\theta}}\frac{\partial(\alpha(\mathbf{q}))}{\partial x} \qquad \Rightarrow \qquad \cos\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial y}=\sin\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

or, equivalently

$$\frac{\partial(\alpha(\mathbf{q}))}{\partial y}=\tan\theta\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

However, from equation (1) we also get

$$\frac{\partial(\alpha(\mathbf{q}))}{\partial y}=-\frac{1}{\tan\theta}\frac{\partial(\alpha(\mathbf{q}))}{\partial x}$$

These two conditions can be satisfied at the same time if and only if $\alpha(\mathbf{q})=0$. We thus conclude that the constraint is nonholonomic.

3. Consider a single-track robot with front fixed wheel and rear steerable wheel. Assuming as configuration vector $\mathbf{q}=[x\ y\ \theta\ \phi]^T$, where $(x\ y\ \theta)$ is the robot pose and ϕ the steering angle, write the kinematic model of the robot without explicitly computing a base of $\mathrm{Null}\left(A^T(\mathbf{q})\right)$.

   If (x,y) represent the position of the rear wheel contact point, and

   $$x_f=x+\ell\cos\theta \qquad y_f=y+\ell\sin\theta \tag{4}$$

   the position of the front wheel contact point, ℓ being the length of the vehicle (distance between the front and rear wheel contact points), we can write the two pure rolling constraints as follows

   $$\dot{x}\sin(\theta+\phi)-\dot{y}\cos(\theta+\phi)=0 \qquad \dot{x}_f\sin\theta-\dot{y}_f\cos\theta=0$$

   From equation (4) a relation between the velocities of the two contact points follows

   $$\dot{x}_f=\dot{x}-\ell\dot{\theta}\sin\theta \qquad \dot{y}_f=\dot{y}+\ell\dot{\theta}\cos\theta$$

   allowing to rewrite the front wheel constraint as

   $$\dot{x}\sin\theta-\dot{y}\cos\theta=\ell\dot{\theta}$$

   We can now write the two constraints together in Pfaffian form

   $$\underbrace{\begin{bmatrix}\sin(\theta+\phi) & -\cos(\theta+\phi) & 0 & 0\\ \sin\theta & -\cos\theta & -\ell & 0\end{bmatrix}}_{A^T(\mathbf{q})}\underbrace{\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix}}_{\dot{\mathbf{q}}}=0$$

   Denoting by $G(\mathbf{q})$ the matrix whose columns are a base of the null space of $A^T(\mathbf{q})$, the kinematic model can be written as follows

   $$\dot{\mathbf{q}}=G(\mathbf{q})\,\mathbf{u}$$

EXERCISE 2

1. What are the approaches that can be used to model the wheel-ground interaction? List them and explain the differences between them.

   There are two main approaches to model the wheel-ground interaction.

   The first one is the empirical approach. In empirical tire models an experimental dataset including lateral forces and corresponding slip angles is assumed to be available, and a class of mathematical functions suitable to fit the dataset is selected. The solution of the fitting problem represents the tire model. A classical example of fitting function is the Pacejka Magic Forrmula.

   The second one is the physical approach. In this case the model that explains the force-slip relation is derived using physical principles. An example of physical model is the brush of Fiala model.

2. The longitudinal force in the brush or Fiala model is given by the following expression

$$F_x = \begin{cases} C_x\sigma_x\left(-1+\frac{|\sigma_x|}{\sigma_{x_{sl}}}-\frac{\sigma_x^2}{3\sigma_{x_{sl}}^2}\right) & |\sigma_x| < \sigma_{x_{sl}} \\ -\mu F_z \text{sign}\left(\sigma_x\right) & |\sigma_x| \geq \sigma_{x_{sl}} \end{cases}$$

   Explain the meaning of each symbol and of the wheel-ground interaction model equation.

   In the brush or Fiala model the tire is modelled as a belt equipped with infinitely many flexible bristles, wrapped around a cylindrical rigid body, which moves on a flat surface. The force is generated by the deformation of the bristles through a linear stiffness model.

   When the slip σ_x is below the minimum slip value that gives full sliding, i.e., $\sigma_{x_{sl}}$, the longitudinal force is given by

$$F_x = C_x\sigma_x\left(-1+\frac{|\sigma_x|}{\sigma_{x_{sl}}}-\frac{\sigma_x^2}{3\sigma_{x_{sl}}^2}\right)$$

   where C_x is the cornering stiffness. When, instead, the slip exceeds $\sigma_{x_{sl}}$, the longitudinal force saturates to the maximum friction force, i.e., $F_x = -\mu F_z \text{sign}\left(\sigma_x\right)$, where μ is the friction coefficient and F_z the normal wheel load.

3. Write the friction circle constraint and explain its meaning.

   In general a tire can generate either longitudinal and lateral forces. In this case, however, we must consider that the total force vector cannot exceed the maximum friction force. If F_x and F_y are the longitudinal and lateral forces, respectively, this constraint can be expressed as

$$\sqrt{F_x^2+F_y^2} \leq \mu F_z$$

   and is called friction circle constraint.

ESERCIZIO 3

1. Define a feasible path planning problem and an optimal path planning problem, stressing the main differences between the two problems.

   A feasible path planning problem is defined as: given a path planning problem $(\mathcal{Q}_{free}, \mathbf{q}_{init}, \mathcal{Q}_{goal})$ find a feasible path (i.e., a collision-free path such that $\sigma(0) = \mathbf{q}_{init}$ and $\sigma(1) \in \mathcal{Q}_{goal}$) $\sigma : [0,1] \to \mathcal{Q}_{free}$ such that $\sigma(0) = \mathbf{q}_{init}$ and $\sigma(1) \in \mathcal{Q}_{goal}$, if one exists; if no such path exists, report failure.

   An optimal path planning problem is defined as: given a path planning problem $(\mathcal{Q}_{free}, \mathbf{q}_{init}, \mathcal{Q}_{goal})$, and a cost function $c : \Sigma \to \mathbb{R}_{\geq 0}$ (where Σ is the set of all paths), find a feasible path $\sigma^\star$ such that $c(\sigma^\star) = \min\{c(\sigma) : \sigma \text{ is feasible}\}$, if one exists; if no such path exists, report failure.

   The main difference is the introduction of a cost function that allows to quantify the "quality" of each path.

2. Describe the algorithm to construct the simplified probabilistic roadmap used by sPRM.

The sPRM algorithm to construct the roadmap follows:

$V \leftarrow \{\mathbf{q}_{init}\} \cup \{SampleFree_i, i = 1, \dots, N\};$
$E \leftarrow \emptyset;$
foreach $\mathbf{v} \in V$ **do**
    $U \leftarrow Near\left(G, \mathbf{v}, r\right) \setminus \{\mathbf{v}\};$
    foreach $\mathbf{u} \in U$ **do**
        if $CollisionFree\left(\mathbf{v}, \mathbf{u}\right)$ **then**
            $E \leftarrow E \cup \{(\mathbf{v}, \mathbf{u})\};$
        end
    end
end
return $G = (V, E)$

N vertex are sampled from the free space, then for each vertex $\mathbf{v}$ the set of near nodes, i.e., the set of nodes in a ball of radius r centred in v, is computed and all the collision free connections between $\mathbf{v}$ and the nodes in the near node set are generated.

3. How the previous algorithm should be modified in order to transform it into its optimal version?

The optimal version of sPRM differs from sPRM only for the near nodes computation, as in the optimal version the radius is computed according to the following rule

$$r = \gamma_{PRM} \left(\log\left(n\right)/n\right)^{1/d}$$

The complete algorithm follows

$V \leftarrow \{\mathbf{q}_{init}\} \cup \{SampleFree_i, i = 1, \dots, N\};$
$E \leftarrow \emptyset;$
foreach $\mathbf{v} \in V$ **do**
    $U \leftarrow Near\left(G, \mathbf{v}, \gamma_{PRM}\left(\log\left(n\right)/n\right)^{1/d}\right) \setminus \{\mathbf{v}\};$
    foreach $\mathbf{u} \in U$ **do**
        if $CollisionFree\left(\mathbf{v}, \mathbf{u}\right)$ **then**
            $E \leftarrow E \cup \{(\mathbf{v}, \mathbf{u})\};$
        end
    end
end
return $G = (V, E)$

ESERCIZIO 4

1. Consider a robot described by the unicycle kinematic model, and a point P related to the unicycle wheel contact point (x, y) by the following relations

$$x_P = x + \varepsilon \cos\theta \qquad y_P = y + \varepsilon \sin\theta$$

Show how a feedback control law that linearises the unicycle model can be derived.

Considering the unicycle model

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \end{aligned}$$

and differentiating the equations of P with respect to time

$$\dot{x}_P = \dot{x} - \varepsilon\dot{\theta}\sin(\theta) = v\cos(\theta) - \varepsilon\dot{\theta}\sin(\theta) = v_{xP}$$
$$\dot{y}_P = \dot{y} + \varepsilon\dot{\theta}\cos(\theta) = v\sin(\theta) + \varepsilon\dot{\theta}\cos(\theta) = v_{yP}$$

Multiplying the two equations by $\cos\theta$ / $\sin\theta$ and summing them together one obtains

$$\begin{array}{rcl} v\cos^2(\theta) - \varepsilon\omega\sin(\theta)\cos(\theta) & = & v_{xP}\cos(\theta) \\ v\sin^2(\theta) + \varepsilon\omega\cos(\theta)\sin(\theta) & = & v_{yP}\sin(\theta) \\ \hline v & = & v_{xP}\cos(\theta) + v_{yP}\sin(\theta) \end{array}$$

Instead, multiplying the two equations by $\sin\theta$ / $\cos\theta$ and subtracting them together yields

$$\begin{array}{rcl} v\cos(\theta)\sin(\theta) - \varepsilon\omega\sin^2(\theta) & = & v_{xP}\sin(\theta) \\ v\sin(\theta)\cos(\theta) + \varepsilon\omega\cos^2(\theta) & = & v_{yP}\cos(\theta) \\ \hline \varepsilon\omega & = & v_{yP}\cos(\theta) - v_{xP}\sin(\theta) \end{array}$$

The change of coordinates that linearises the unicycle model is thus given by

$$v = v_{xP}\cos(\theta) + v_{yP}\sin(\theta)$$
$$\omega = \frac{v_{yP}\cos(\theta) - v_{xP}\sin(\theta)}{\varepsilon}$$

2. Design a trajectory tracking controller based on the linearising law introduced in the previous step, and draw a block diagram of the entire control system, explaining how it can be tuned.

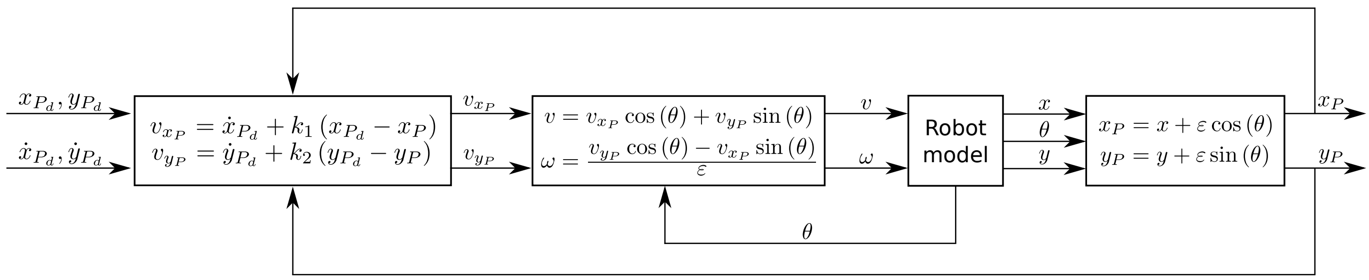
The result of the feedback linearising controller is a dynamical system composed of two independent integrators

$$\dot{x}_P = v_{xP}$$
$$\dot{y}_P = v_{yP}$$

A simple PD controller can be thus used to track the desired trajectory

$$v_{xP} = \dot{x}_{P_d} + k_1(x_{P_d} - x_P)$$
$$v_{yP} = \dot{y}_{P_d} + k_2(y_{P_d} - y_P)$$

A diagram of the entire control system is shown in the following figure



Moreover, $k = k_1 = k_2$ can be selected in order to enforce the desired trajectory tracking error convergence rate. In fact, the convergence is exponential and k represents the eigenvalue of the error dynamics.

3. Explain why the control system designed in the previous step cannot control the robot heading.

Due to the feedback linearising controller the robot orientation becomes an hidden state, and cannot be controlled any more. For this reason that control system cannot be used to control the robot heading but only its position.