Control of Mobile Robots
Exercise 1: Kinematics
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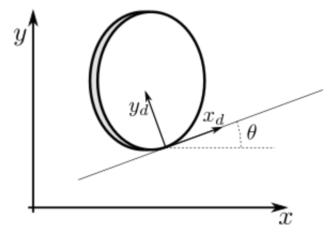
Exercise 1 - Pure rolling disk

Consider a disk rolling without slipping on the horizontal plane (x-y plane), keeping the sagittal plane in the vertical direction (z direction).
Write the kinematic constraint in Pfaffian form at which the disk is subjected to.



Solution

The rolling disk configuration is represented by vector $\mathbf{q} = [x,\ y,\ \theta]^T$, and the disk is subjected to a kinematic constraint that, in the local disk reference frame (x_d, y_d), can be expressed as $\dot{y}_d = 0$ (only the velocities parallel to x_d are admissible).
In order to write this constraint as a kinematic constraint in Pfaffian form, it must be expressed in terms of the configuration variables, and thus in the global reference frame (x, y).
The local disk reference frame is related to the global reference frame by the following rotation matrix

$$R_d = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Velocities in the global reference frame are thus related to velocities in the local disk frame by the relation

$$\begin{bmatrix} \dot{x}_d \\ \dot{y}_d \end{bmatrix} = R_d^T \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{x}\cos\theta + \dot{y}\sin\theta \\ \dot{y}\cos\theta - \dot{x}\sin\theta \end{bmatrix}$$

The constraint $\dot{y}_d = 0$ can be thus rewritten in terms of the configuration variables as

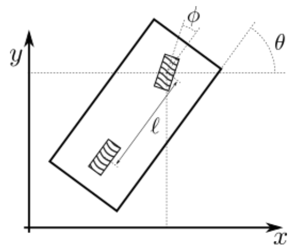
$$\dot{y}\cos\theta - \dot{x}\sin\theta = \dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

or in Pfaffian form as

$$\begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0$$

Exercise 2 - Bicycle robot

Write the kinematic model of a bicycle with front steerable wheel, considering as configuration vector $\mathbf{q} = [x,\ y,\ \theta,\ \phi]^T$, where $x,\ y$ denote the position of the front wheel.



Solution

The two wheels are subjected to pure rolling constraints

$$\begin{aligned}\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) = 0\\ \dot{x}_r\sin(\theta) - \dot{y}_r\cos(\theta) = 0\end{aligned}$$

where (x_r, y_r) is the position of the rear wheel contact point, and it is related to the position of the front wheel contact point (x, y) through a rigidity constraint

$$\begin{aligned}x_r &= x - \ell\cos\theta\\ y_r &= y - \ell\sin\theta\end{aligned}$$

Differentiating the rigidity constraints with respect to time we obtain

$$\begin{aligned}\dot{x}_r &= \dot{x} + \ell\dot{\theta}\sin\theta\\ \dot{y}_r &= \dot{y} - \ell\dot{\theta}\cos\theta\end{aligned}$$

Using these two relations we can write the rear wheel constraint in terms of the configuration variables as follows

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) + \ell\dot{\theta} = 0$$

The two constraints that describe the bicycle can be written in Pfaffian form as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix}\sin(\theta+\phi) & -\cos(\theta+\phi) & 0 & 0\\ \sin(\theta) & -\cos(\theta) & \ell & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} = 0 \qquad (1)$$

To determine the bicycle kinematic model we need to solve this set of equations for $\dot{\mathbf{q}}$, i.e., we need to compute a basis for the null space of matrix $A^T(\mathbf{q})$. Using a symbolic manipulation tool we obtain the following basis for the null space of matrix $A^T(\mathbf{q})$

$$\left\{\begin{bmatrix}\cos(\theta+\phi)\\ \sin(\theta+\phi)\\ \dfrac{\sin(\phi)}{\ell}\\ 0\\ 0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix},\right\}$$

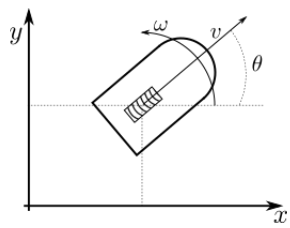
The kinematic model of the bicycle is given by

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}\cos(\theta+\phi)\\ \sin(\theta+\phi)\\ \dfrac{\sin(\phi)}{\ell}\\ 0\end{bmatrix} v + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\omega$$

where v is the velocity of the front wheel, and ω the steering velocity of the front wheel.

Exercise 3 - Unicycle robot

Consider a unicycle robot whose pose is described by the wheel contact point (x, y) and the angle θ.
Write the kinematic constraint at which the unicycle robot is subjected to, and show that it is a nonholonomic constraint.



Solution

The unicycle robot configuration is represented by vector $\mathbf{q} = [x,\ y,\ \theta]^T$, and its wheel is subjected to a pure rolling constraint

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

Writing the rolling constraint in Pfaffian form we have

$$\begin{bmatrix}\sin\theta & -\cos\theta & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = 0$$

and we define $X(\mathbf{q}) = \sin\theta$, $Y(\mathbf{q}) = -\cos\theta$, and $\Theta(\mathbf{q}) = 0$.
In order to show that this is a nonholonomic constraint we can apply the necessary and sufficient condition. We should find a function $\alpha(\mathbf{q})$ such that

$$\frac{\partial(\alpha(\mathbf{q})X(\mathbf{q}))}{\partial y} = \frac{\partial(\alpha(\mathbf{q})Y(\mathbf{q}))}{\partial x}$$
$$\frac{\partial(\alpha(\mathbf{q})X(\mathbf{q}))}{\partial \theta} = \frac{\partial(\alpha(\mathbf{q})\Theta(\mathbf{q}))}{\partial x}$$
$$\frac{\partial(\alpha(\mathbf{q})Y(\mathbf{q}))}{\partial \theta} = \frac{\partial(\alpha(\mathbf{q})\Theta(\mathbf{q}))}{\partial y}$$

and substituting the definitions of $X(\mathbf{q})$, $Y(\mathbf{q})$, $\Theta(\mathbf{q})$ we get

$$\frac{\partial\alpha(\mathbf{q})}{\partial y}\sin\theta = -\frac{\partial\alpha(\mathbf{q})}{\partial x}\cos\theta \tag{2}$$
$$\frac{\partial\alpha(\mathbf{q})}{\partial \theta}\sin\theta + \alpha(\mathbf{q})\cos\theta = 0 \tag{3}$$
$$-\frac{\partial\alpha(\mathbf{q})}{\partial \theta}\cos\theta + \alpha(\mathbf{q})\sin\theta = 0 \tag{4}$$

From relation (4) we obtain

$$\frac{\partial\alpha(\mathbf{q})}{\partial \theta} = \alpha(\mathbf{q})\tan\theta$$

and introducing this result into (3)

$$\alpha(\mathbf{q})\sin^2\theta + \alpha(\mathbf{q})\cos^2\theta = 0 \qquad \Rightarrow \qquad \alpha(\mathbf{q}) = 0$$

We thus conclude that the constraint is nonholonomic.

Exercise 4 - Kinematic constraints and kinematic model

Consider a mobile robot whose configuration is described by vector $\mathbf{q} = [q_1,\ q_2,\ q_3,\ q_4]^T$, and characterized by the following two constraints

$$\begin{aligned} \dot{q}_1 + q_1\dot{q}_2 + \dot{q}_3 &= 0 \\ \dot{q}_2 + q_2\dot{q}_3 &= 0 \end{aligned}$$

Does the following kinematic model

$$\dot{\mathbf{q}} = \begin{bmatrix} q_1q_2 - 1 \\ -q_2 \\ 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

describe the motion of the robot?

Solution

The two constraints can be represented in Pfaffian form as

$$A^T(\mathbf{q})\dot{\mathbf{q}} = \begin{bmatrix} 1 & q_1 & 1 & 0 \\ 0 & 1 & q_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = 0$$

If the given equation is the kinematic model for a robot described by these two constraints, the two vectors

$$\begin{bmatrix} q_1q_2 - 1 \\ -q_2 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

should belong to the null space of $A^T(\mathbf{q})$ and be linearly independent. This can be easily verified showing that

$$A^T(\mathbf{q}) \begin{bmatrix} q_1q_2 - 1 \\ -q_2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0} \qquad A^T(\mathbf{q}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

and that

$$\text{rank}\left(\begin{bmatrix} q_1q_2 - 1 & 0 \\ -q_2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 2$$

Exercise 5 - System of kinematic constraints

Consider the following systems of kinematic constraints

$$\begin{aligned}\dot{q}_1 + q_1\dot{q}_2 + \dot{q}_3 &= 0\\ \dot{q}_1 + \dot{q}_2 + q_1\dot{q}_3 &= 0\end{aligned}$$

where $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T$ and $q_1 \neq 1$, and

$$\begin{aligned}\dot{q}_1 + q_1\dot{q}_2 + 6\dot{q}_3 &= 0\\ 4\dot{q}_2 + q_2\dot{q}_3 &= 0\end{aligned}$$

where $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T$.
Determine if they are holonomic or nonholonomic.

Solution

Considering the first set, constraints can be rewritten in Pfaffian form as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} 1 & q_1 & 1 \\ 1 & 1 & q_1 \end{bmatrix} \dot{\mathbf{q}} = 0$$

Under the assumption $q_1 \neq 1$, taking the first and last columns it can be easily verified that $\text{rank}\left(A^T(\mathbf{q})\right) = 2$.
A basis of the null space of $A^T(\mathbf{q})$ is composed by a single vector

$$g_1(\mathbf{q}) = \begin{bmatrix} 1 + q_1 \\ -1 \\ -1 \end{bmatrix}$$

The procedure to compute the accessibility distribution is initialized with $\Delta_1 = \text{span}\{g_1\}$, but no other vector fields can be added. We conclude that the accessibility space has dimension 1, that is equal to $n - k$, and thus the system of constraints is holonomic.
Considering the second set, constraints can be rewritten in Pfaffian form as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} 1 & q_1 & 6 & 0 \\ 0 & 4 & q_2 & 0 \end{bmatrix} \dot{\mathbf{q}} = 0$$

Taking the first two columns, it can be easily verified that $\text{rank}\left(A^T(\mathbf{q})\right) = 2$.
A basis of the null space of $A^T(\mathbf{q})$ is composed by the following two vectors

$$g_1(\mathbf{q}) = \begin{bmatrix} q_1q_2 - 24 \\ -q_2 \\ 4 \\ 0 \end{bmatrix} \qquad g_2(\mathbf{q}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The procedure to compute the accessibility distribution is initialized with $\Delta_1 = \text{span}\{g_1, g_2\}$.
A third vector can be generated as

$$g_3(\mathbf{q}) = [g_1, g_2] = \frac{\partial g_2}{\partial \mathbf{q}} g_1 - \frac{\partial g_1}{\partial \mathbf{q}} g_2 = \mathbf{0} - \begin{bmatrix} q_2 & q_1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

No other vector fields can be added, we conclude that the accessibility space has dimension 2, that is equal to $n - k$, and thus the system of constraints is holonomic.

Exercise 6 - System of kinematic constraints

Consider the following system of kinematic constraints

$$q_1^2\dot{q}_2 + (1-q_1)\,\dot{q}_3 + \dot{q}_4 = 0$$
$$6\dot{q}_1 + (1-q_1)\,\dot{q}_2 + 4\dot{q}_3 = 0$$

where $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^T$.

Answer to the following questions:

1. is each constraint, considered as an independent constraint, holonomic or nonholonomic?
2. is the system of constraints holonomic or nonholonomic?

Solution

Considering the first constraint

$$a^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} 0 & q_1^2 & 1-q_1 & 1 \end{bmatrix} \dot{\mathbf{q}} = 0$$

as an independent constraint, we can write the following equalities

$$\frac{\partial\left[\alpha(\mathbf{q})\,a_k(\mathbf{q})\right]}{\partial q_j} = \frac{\partial\left[\alpha(\mathbf{q})\,a_j(\mathbf{q})\right]}{\partial q_k}$$

for $(j,k) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$, obtaining

$$\frac{\partial\left[\alpha(\mathbf{q})\,q_1^2\right]}{\partial q_1} = q_1^2\frac{\partial\alpha(\mathbf{q})}{\partial q_1} + 2q_1\alpha(\mathbf{q}) = \frac{\partial\left[\alpha(\mathbf{q})\cdot 0\right]}{\partial q_2} = 0$$
$$\frac{\partial\left[\alpha(\mathbf{q})\,(1-q_1)\right]}{\partial q_1} = (1-q_1)\,\frac{\partial\alpha(\mathbf{q})}{\partial q_1} - \alpha(\mathbf{q}) = \frac{\partial\left[\alpha(\mathbf{q})\cdot 0\right]}{\partial q_3} = 0$$
$$\frac{\partial\alpha(\mathbf{q})}{\partial q_1} = \frac{\partial\left[\alpha(\mathbf{q})\cdot 0\right]}{\partial q_4} = 0$$
$$\frac{\partial\left[\alpha(\mathbf{q})\,(1-q_1)\right]}{\partial q_2} = (1-q_1)\,\frac{\partial\alpha(\mathbf{q})}{\partial q_2} = \frac{\partial\left[\alpha(\mathbf{q})\,q_1^2\right]}{\partial q_3} = q_1^2\frac{\partial\alpha(\mathbf{q})}{\partial q_3}$$
$$\frac{\partial\alpha(\mathbf{q})}{\partial q_2} = \frac{\partial\left[\alpha(\mathbf{q})\,q_1^2\right]}{\partial q_4} = q_1^2\frac{\partial\alpha(\mathbf{q})}{\partial q_4}$$
$$\frac{\partial\alpha(\mathbf{q})}{\partial q_3} = \frac{\partial\left[\alpha(\mathbf{q})\,(1-q_1)\right]}{\partial q_4} = (1-q_1)\,\frac{\partial\alpha(\mathbf{q})}{\partial q_4}$$

From the third equation it follows that

$$\frac{\partial\alpha(\mathbf{q})}{\partial q_1} = 0$$

Introducing this relation into the first equation we obtain

$$2q_1\alpha(\mathbf{q}) = 0$$

and thus the only solution is $\alpha(\mathbf{q}) = 0$. As a consequence, the first is a nonholonomc constraint.
Considering now the second constraint

$$a^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} 6 & 1-q_1 & 4 & 0 \end{bmatrix} \dot{\mathbf{q}} = 0$$

as an independent constraint, we can write the following equalities

$$\frac{\partial\left[\alpha(\mathbf{q})\,a_k(\mathbf{q})\right]}{\partial q_j} = \frac{\partial\left[\alpha(\mathbf{q})\,a_j(\mathbf{q})\right]}{\partial q_k}$$

for $(j,k) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$, obtaining

$$\begin{aligned}
&\frac{\partial\left[\alpha(\mathbf{q})(1-q_1)\right]}{\partial q_1} = (1-q_1)\frac{\partial\alpha(\mathbf{q})}{\partial q_1} - \alpha(\mathbf{q}) = \frac{\partial\left[6\alpha(\mathbf{q})\right]}{\partial q_2} = 6\frac{\partial\alpha(\mathbf{q})}{\partial q_2}\\
&\frac{\partial\left[4\alpha(\mathbf{q})\right]}{\partial q_1} = 4\frac{\partial\alpha(\mathbf{q})}{\partial q_1} = \frac{\partial\left[6\alpha(\mathbf{q})\right]}{\partial q_3} = 6\frac{\partial\alpha(\mathbf{q})}{\partial q_3}\\
&\frac{\partial\left[\alpha(\mathbf{q})\cdot 0\right]}{\partial q_1} = 0 = \frac{\partial\left[6\alpha(\mathbf{q})\right]}{\partial q_4} = 6\frac{\partial\alpha(\mathbf{q})}{\partial q_4}\\
&\frac{\partial\left[4\alpha(\mathbf{q})\right]}{\partial q_2} = 4\frac{\partial\alpha(\mathbf{q})}{\partial q_2} = \frac{\partial\left[\alpha(\mathbf{q})(1-q_1)\right]}{\partial q_3} = (1-q_1)\frac{\partial\alpha(\mathbf{q})}{\partial q_3}\\
&\frac{\partial\left[\alpha(\mathbf{q})\cdot 0\right]}{\partial q_2} = 0 = \frac{\partial\left[\alpha(\mathbf{q})(1-q_1)\right]}{\partial q_4} = (1-q_1)\frac{\partial\alpha(\mathbf{q})}{\partial q_4}\\
&\frac{\partial\left[\alpha(\mathbf{q})\cdot 0\right]}{\partial q_3} = 0 = \frac{\partial\left[4\alpha(\mathbf{q})\right]}{\partial q_4} = 4\frac{\partial\alpha(\mathbf{q})}{\partial q_4}
\end{aligned}$$

From the third, fifth and sixth equations it follows that

$$\frac{\partial\alpha(\mathbf{q})}{\partial q_4} = 0$$

Instead, combining second and fourth equations we obtain

$$\begin{aligned}
\frac{\partial\alpha(\mathbf{q})}{\partial q_2} &= \frac{1-q_1}{4}\frac{\partial\alpha(\mathbf{q})}{\partial q_3}\\
\frac{\partial\alpha(\mathbf{q})}{\partial q_1} &= \frac{3}{2}\frac{\partial\alpha(\mathbf{q})}{\partial q_3}
\end{aligned}$$

and from them

$$\frac{\partial\alpha(\mathbf{q})}{\partial q_1} = \frac{3}{2}\frac{\partial\alpha(\mathbf{q})}{\partial q_3} = \frac{3}{2}\frac{4}{1-q_1}\frac{\partial\alpha(\mathbf{q})}{\partial q_2} = \frac{6}{1-q_1}\frac{\partial\alpha(\mathbf{q})}{\partial q_2}$$

Finally, including this relation into the first equation we obtain

$$\alpha(\mathbf{q}) + 6\frac{\partial\alpha(\mathbf{q})}{\partial q_2} = (1-q_1)\frac{6}{1-q_1}\frac{\partial\alpha(\mathbf{q})}{\partial q_2} = 6\frac{\partial\alpha(\mathbf{q})}{\partial q_2}$$

that results into $\alpha(\mathbf{q}) = 0$.
As a consequence, the second constraint is nonholonomic.
Consider now the system of two constraints, the set can be rewritten in Pfaffian form as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} 0 & q_1^2 & 1-q_1 & 1\\ 6 & 1-q_1 & 4 & 0 \end{bmatrix}\dot{\mathbf{q}} = 0$$

Taking the first and last columns, it can be easily verified that $\operatorname{rank}\left(A^T(\mathbf{q})\right) = 2$.
A basis of the null space of $A^T(\mathbf{q})$ is composed by the two vectors

$$g_1(\mathbf{q}) = \begin{bmatrix} 0\\ -4\\ 1-q_1\\ 3q_1^2+2q_1-1 \end{bmatrix} \qquad g_2(\mathbf{q}) = \begin{bmatrix} -4\\ 0\\ 6\\ 6(q_1-1) \end{bmatrix}$$

The procedure to compute the accessibility distribution is initialized with $\Delta_1 = \operatorname{span}\{g_1, g_2\}$.
To construct Δ_2 we have to add to Δ_1 the vector fields obtained by the Lie bracket of all possible combinations of the elements of Δ_1, that are linearly independent with respect to g_1 and g_2. The only available combination is g_1, g_2 giving rise to

$$\begin{aligned}
g_3(\mathbf{q}) = [g_1, g_2] = \frac{\partial g_2}{\partial\mathbf{q}}g_1 - \frac{\partial g_1}{\partial\mathbf{q}}g_2 &= \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 6&0&0&0 \end{bmatrix}\begin{bmatrix} 0\\ -4\\ 1-q_1\\ 3q_1^2+2q_1-1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ -1&0&0&0\\ 6q_1+2&0&0&0 \end{bmatrix}\begin{bmatrix} -4\\ 0\\ 6\\ 6(q_1-1) \end{bmatrix}\\
&= \begin{bmatrix} 0\\ 0\\ -4\\ 8(3q_1+1) \end{bmatrix}
\end{aligned}$$

as a consequence $\Delta_2 = \text{span}\{g_1, g_2, g_3\}$.
Again to construct Δ_3 we have to add to Δ_2 the vector fields obtained by the Lie bracket of all possible combinations including one element of Δ_2 and one of Δ_1, that are linearly independent with respect to g_1, g_2 and g_3. The available candidates are (already excluding Brackets that are equal to 0, e.g., $[g_1, g_1]$, and Brackets that are equal except for the sign, e.g. $[g_1, g_2]$ and $[g_2, g_1]$)

$$[g_1, g_2] \quad [g_1, g_3] \quad [g_2, g_3]$$

Excluding the first one that is already in Δ_2 as g_3, we can compute the second and the third

$$[g_1, g_3] = \frac{\partial g_3}{\partial \mathbf{q}} g_1 - \frac{\partial g_1}{\partial \mathbf{q}} g_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -4 \\ 1 - q_1 \\ 3q_1^2 + 2q_1 - 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 6q_1 + 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -4 \\ 8\left(3q_1 + 1\right) \end{bmatrix} = 0$$

$$[g_2, g_3] = \frac{\partial g_3}{\partial \mathbf{q}} g_2 - \frac{\partial g_2}{\partial \mathbf{q}} g_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 6 \\ 6\left(q_1 - 1\right) \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -4 \\ 8\left(3q_1 + 1\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 96 \end{bmatrix}$$

We exclude the first as it is the null vector and keep the second defining

$$g_4\left(\mathbf{q}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 96 \end{bmatrix}$$

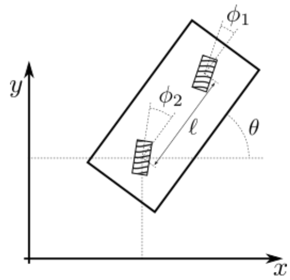
Finally, $\Delta_3 = \text{span}\{g_1, g_2, g_3, g_4\}$.
As $n = 4$ and $m = 2$, from the theory we now that $\Delta_A = \Delta_{n-m+1} = \Delta_3$. Consequently, we only need to verify that all the four vectors are linearly independent

$$\det\left(\begin{bmatrix} 0 & -4 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 1 - q_1 & 6 & -4 & 0 \\ 3q_1^2 + 2q_1 - 1 & 6\left(q_1 - 1\right) & 8\left(3q_1 + 1\right) & 96 \end{bmatrix}\right) = 6144$$

We conclude that the system of constraints is nonholonomic.

Exercise 7 - Bicycle robot

Write the kinematic model of a bicycle with front and rear steerable wheels.



Solution

The bicycle robot configuration is represented by vector

$$\mathbf{q} = [x,\ y,\ \theta,\ \phi_1,\ \phi_2]^T$$

and the two wheels are subjected to pure rolling constraints

$$\dot{x}_1 \sin(\theta + \phi_1) - \dot{y}_1 \cos(\theta + \phi_1) = 0$$
$$\dot{x} \sin(\theta + \phi_2) - \dot{y} \cos(\theta + \phi_2) = 0$$

where (x_1, y_1) is the position of the front wheel contact point, and it is related to the position of the rear wheel contact point (x, y) through a rigidity constraint

$$x_1 = x + \ell \cos\theta$$
$$y_1 = y + \ell \sin\theta$$

Differentiating the rigidity constraint with respect to time we obtain

$$\dot{x}_1 = \dot{x} - \ell\dot{\theta}\sin\theta$$
$$\dot{y}_1 = \dot{y} + \ell\dot{\theta}\cos\theta$$

Using these two relations we can write the front wheel constraint in terms of the configuration variables as follows

$$\dot{x}\sin(\theta + \phi_1) - \dot{y}\cos(\theta + \phi_1) - \ell\dot{\theta}\cos\phi_1 = 0$$

The two constraints that describe the bicycle can be written in Pfaffian form as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} \sin(\theta+\phi_1) & -\cos(\theta+\phi_1) & -\ell\cos\phi_1 & 0 & 0 \\ \sin(\theta+\phi_2) & -\cos(\theta+\phi_2) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = 0 \tag{5}$$

To determine the bicycle kinematic model we need to solve this set of equations for $\dot{\mathbf{q}}$, i.e., we need to compute a basis for the null space of matrix $A^T(\mathbf{q})$.
To simplify the solution, we can use the "reduced row echelon" form of matrix $A^T(\mathbf{q})$, applying the following row operations (in the following, $\mathbf{r}_1$ and $\mathbf{r}_2$ denotes the first and the second row of $A^T(\mathbf{q})$):

1. After applying the row operations $\mathbf{r}_1 \leftarrow \mathbf{r}_1 - \mathbf{r}_2\dfrac{\cos(\theta+\phi_1)}{\cos(\theta+\phi_2)}$ and $\mathbf{r}_1 \leftarrow \mathbf{r}_1\cos(\theta+\phi_2)$, $\mathbf{r}_1$ becomes

$$\mathbf{r}_1 = [\sin(\phi_1 - \phi_2) \quad 0 \quad -l\cos(\phi_1)\cos(\theta+\phi_2) \quad 0 \quad 0]$$

2. To make the first element of $\mathbf{r}_1$ equal to 1, we can apply $\mathbf{r}_1 \leftarrow \dfrac{\mathbf{r}_1}{\sin(\phi_1 - \phi_1)}$ obtaining

$$\mathbf{r}_1 = \begin{bmatrix} 1 & 0 & \dfrac{-l\cos(\phi_1)\cos(\theta+\phi_2)}{\sin(\phi_1-\phi_2)} & 0 & 0 \end{bmatrix}$$

3. Applying the row operation $\mathbf{r}_2 \leftarrow \mathbf{r}_2 - \mathbf{r}_1\sin(\theta+\phi_2)$, the first element of $\mathbf{r}_2$ becomes equal to zero

$$\mathbf{r}_2 = \begin{bmatrix} 0 & -\cos(\theta+\phi_2) & \dfrac{l\cos(\phi_1)\cos(\theta+\phi_2)\sin(\theta+\phi_2)}{\sin(\phi_1-\phi_2)} & 0 & 0 \end{bmatrix}$$

4. Finally, applying the row operation $\mathbf{r}_2 \leftarrow -\dfrac{\mathbf{r}_2}{\cos(\theta+\phi 2)}$, $\mathbf{r}_2$ becomes

$$\mathbf{r}_2 = \begin{bmatrix} 0 & 1 & \dfrac{-l\cos(\phi_1)\sin(\theta+\phi_2)}{\sin(\phi_1-\phi_2)} & 0 & 0 \end{bmatrix}$$

The augmented matrix is then written as

$$\left[\begin{array}{ccccc|c} 1 & 0 & \dfrac{-l\cos(\phi_1)\cos(\theta+\phi_2)}{\sin(\phi_1-\phi_2)} & 0 & 0 & 0 \\ 0 & 1 & \dfrac{-l\cos(\phi_1)\sin(\theta+\phi_2)}{\sin(\phi_1-\phi_2)} & 0 & 0 & 0 \end{array}\right]$$

which corresponds to the following system of equations,

$$\begin{aligned} \dot{x} - \frac{l\cos(\phi_1)\cos(\theta+\phi_2)}{\sin\phi_1-\phi_2}\dot{\theta} &= 0 \\ \dot{y} - \frac{l\cos(\phi_1)\sin(\theta+\phi_2)}{\sin\phi_1-\phi_2}\dot{\theta} &= 0 \end{aligned}$$

As we have five variables and two equations, we can select the remaining three variables freely. Let u_1, u_2, and u_3 denote the free variables, we can select

$$\begin{aligned} u_1 &= \dot{\theta} \\ u_2 &= \dot{\phi}_1 = w_1 \\ u_3 &= \dot{\phi}_2 = w_2. \end{aligned}$$

A solution to the equation (5) can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{l\cos(\phi_1)\cos(\theta+\phi_2)}{\sin\phi_1-\phi_2} \\ \dfrac{l\cos(\phi_1)\sin(\theta+\phi_2)}{\sin\phi_1-\phi_2} \\ 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_3$$

which corresponds to the kinematic model with inputs $\dot{\theta}$, w_1, and w_2.
A basis for the null space of matrix $A^T(\mathbf{q})$ is thus given by

$$\left\{ \begin{bmatrix} \dfrac{\ell\cos\phi_1\cos(\theta+\phi_2)}{\sin(\phi_1-\phi_2)} \\ \dfrac{\ell\cos\phi_1\sin(\theta+\phi_2)}{\sin(\phi_1-\phi_2)} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

In order to select, among all the possible basis, one that allows to have inputs to the kinematic model that can be easily interpreted as physical quantities, we can multiply the first vector by $\dfrac{\sin(\phi_1-\phi_2)}{\ell\cos\phi_1}$ obtaining

$$\left\{ \begin{bmatrix} \cos(\theta+\phi_2) \\ \sin(\theta+\phi_2) \\ \dfrac{\sin(\phi_1-\phi_2)}{\ell\cos\phi_1} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

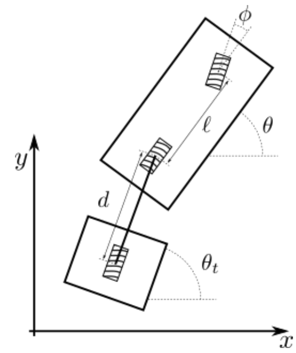
The kinematic model of the bicycle is given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi_2) \\ \sin(\theta+\phi_2) \\ \dfrac{\sin(\phi_1-\phi_2)}{\ell\cos\phi_1} \\ 0 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \omega_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega_2$$

where v is the velocity of the rear wheel and ω_1, ω_2 the steering velocities for the front and rear wheels.

Exercise 8 - Car-trailer system

Write the kinematic model of a car-trailer system.
The car position (x, y) is represented by the rear wheel contact point, the trailer position (x_t, y_t), instead, is represented by the trailer wheel contact point.



Solution

The car-trailer robot configuration is represented by vector

$$\mathbf{q} = [x,\ y,\ \theta,\ \theta_t,\ \phi]^T$$

and we can write the pure rolling constraints referred to each wheel of the car and of the trailer as follows

$$\dot{x}_1 \sin(\theta + \phi) - \dot{y}_1 \cos(\theta + \phi) = 0 \tag{6}$$
$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0 \tag{7}$$
$$\dot{x}_t \sin\theta_t - \dot{y}_t \cos\theta_t = 0 \tag{8}$$

where (x_1, y_1) is the position of the car front wheel contact point.
We can relate the position of the front wheel contact point and of the trailer wheel contact point to (x, y) through a rigidity constraint

$$x_1 = x + \ell \cos\theta$$
$$y_1 = y + \ell \sin\theta$$

and

$$x_t = x - d \cos\theta_t$$
$$y_t = y - d \sin\theta_t$$

Differentiating the two relations with respect to time we obtain

$$\dot{x}_1 = \dot{x} - \ell\dot{\theta} \sin\theta$$
$$\dot{y}_1 = \dot{y} + \ell\dot{\theta} \cos\theta$$

and

$$\dot{x}_t = \dot{x} + d\dot{\theta}_t \sin\theta_t$$
$$\dot{y}_t = \dot{y} - d\dot{\theta}_t \cos\theta_t$$

Substituting these relations in (6) and (8) we obtain

$$\dot{x} \sin(\theta + \phi) - \dot{y} \cos(\theta + \phi) - \ell\dot{\theta} \cos\phi = 0$$
$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0$$
$$\dot{x} \sin\theta_t - \dot{y} \cos\theta_t + d\dot{\theta}_t = 0$$

The three constraints that describe the car-trailer can be written in Pfaffian form as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} \sin(\theta + \phi) & -\cos(\theta + \phi) & -\ell \cos\phi & 0 & 0 \\ \sin\theta & -\cos\theta & 0 & 0 & 0 \\ \sin\theta_t & -\cos\theta_t & 0 & d & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\theta}_t \\ \dot{\phi} \end{bmatrix} = 0$$

A basis for the null space of $A^T(\mathbf{q})\,\dot{\mathbf{q}}$ is given by

$$\left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{\ell}\tan\phi \\ \frac{1}{d}\sin(\theta - \theta_t) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Finally, the kinematic model for the car-trailer is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\theta}_t \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \dfrac{1}{\ell}\tan\phi \\ \dfrac{1}{d}\sin(\theta - \theta_t) \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega \tag{9}$$

where v is the velocity of the rear wheel of the car, and ω the steering velocity for the front wheel.