Automatic Control
Exercise 4: Frequency domain design
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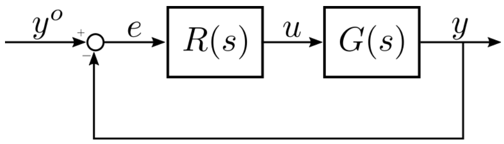
Exercise 1

Consider the following control system



where $G(s) = \dfrac{2}{(1+10s)^2(1+0.1s)}$.
Compute the transfer function $R(s)$ of a controller in such a way that:

- $|e_\infty| \leq 0.1$ for $y^o(t) = sca(t)$;
- the phase margin φ_m is greater or equal to 80^o;
- the crossover frequency ω_c is greater or equal to $0.1\, rad/s$.

Solution

Steady-state design
We start from the steady-state design, assuming that once the design will be completed the closed-loop system will be asymptotically stable.
The steady-state error due to the reference is

$$e_\infty = \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{1}{s} = \lim_{s\to 0}\frac{1}{1+\frac{2\mu_r}{s^{g_r}}} = \lim_{s\to 0}\frac{s^{g_r}}{s^{g_r}+2\mu_r} = \frac{1}{1+2\mu_r} \qquad g_r = 0$$

Enforcing the steady-state requirements we obtain

$$\frac{1}{1+2\mu_r} \leq 0.1 \qquad \Rightarrow \qquad \mu_r \geq 4.5$$

We complete the steady-state design assuming $\mu_r = 5$, thus

$$R_1(s) = 5$$

Transient design
We start the transient design with

$$L_1(s) = \frac{10}{(1+10s)^2(1+0.1s)}$$

Magnitude Bode diagram of $L_1(s)$ is shown in Fig. 1.

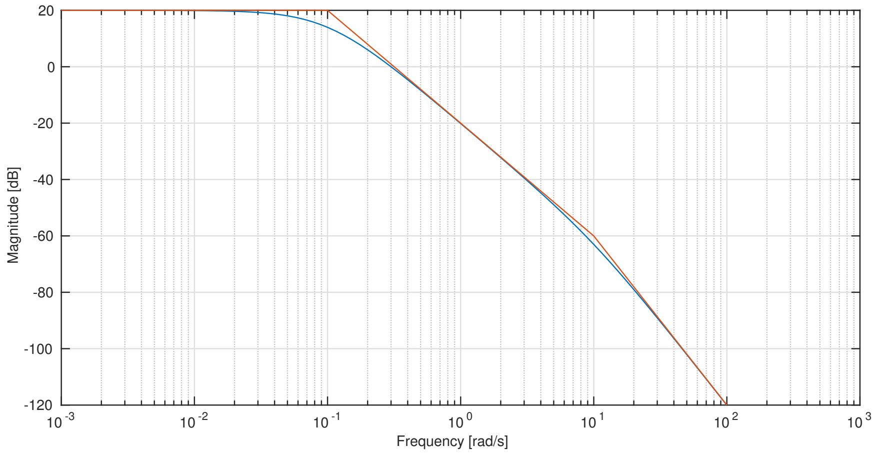


Figure 1: $L_1(s)$ magnitude Bode diagram.

Though the crossover frequency fulfils the requirements, having slope -2 at the crossover the phase margin cannot fulfil it, we thus need to revise the design.
The loop transfer function has to be reshaped in such a way that it has the same gain and type of $L_1(s)$ (same low frequency behaviour) and it has slope at least -3 at high frequency (regulator causality). As $G(s)$ is the transfer function of a minimum phase system we decide to cross the $0\,dB$-axis with slope -1 at $0.1\,rad/s$, the minimum crossover frequency that satisfies the requirement specifications. Doing in this way we obtain the loop transfer function depicted in Fig. 2.

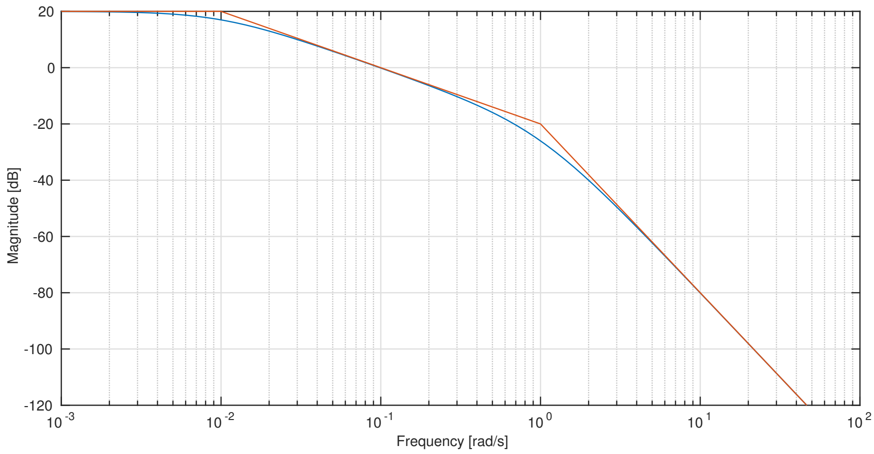


Figure 2: $L_a(s)$ magnitude Bode diagram, first solution.

The corresponding loop transfer function is

$$L_a(s) = \frac{10}{(1+100s)(1+s)^2}$$

which is characterised by

$$\omega_c = 0.1\,rad/s$$
$$\varphi_c = -\arctan(100 \cdot 0.1) - 2\arctan(0.1) \approx -95.7^o$$

and finally

$$\varphi_m = 180^o - |\varphi_c| \approx 84.3^o$$

The requirements on crossover frequency and phase margin are both fulfilled.
The corresponding regulator is

$$R_a(s) = \frac{L_a(s)}{G(s)} = \frac{10}{(1+100s)(1+s)^2}\,\frac{(1+10s)^2(1+0.1s)}{2} = 5\frac{(1+10s)^2(1+0.1s)}{(1+100s)(1+s)^2}$$

Following the same reasoning, but taking as crossover frequency $1\,rad/s$, we obtain the loop transfer function depicted in Fig. 3.

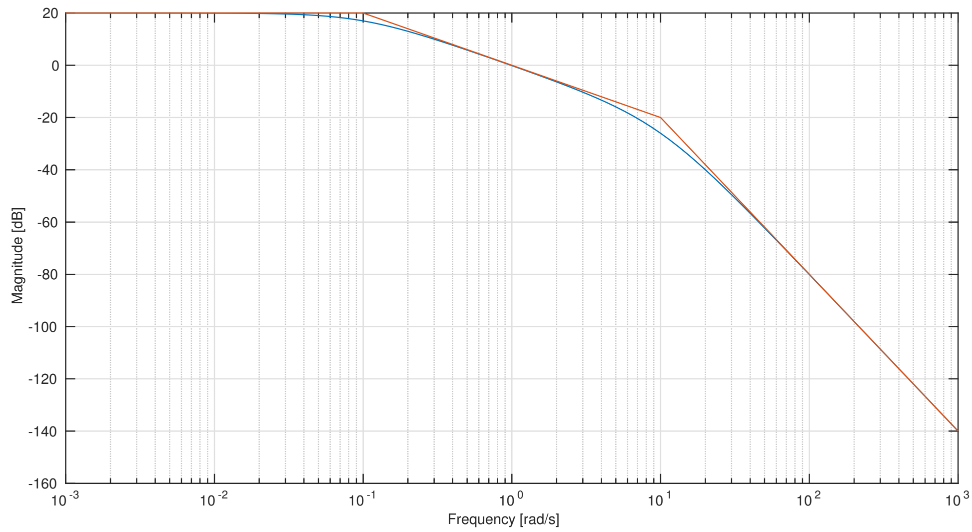


Figure 3: $L_b(s)$ magnitude Bode diagram, second solution.

The corresponding loop transfer function is

$$L_b(s) = \frac{10}{(1+10s)(1+0.1s)^2}$$

which is characterised by

$$\omega_c = 1\,rad/s$$
$$\varphi_c = -\arctan(10) - 2\arctan(0.1) \approx -95.7^o$$

and finally

$$\varphi_m = 180^o - |\varphi_c| \approx 84.3^o$$

The requirements on crossover frequency and phase margin are both fulfilled.
The corresponding regulator is

$$R_b(s) = \frac{L_b(s)}{G(s)} = \frac{10}{(1+10s)(1+0.1s)^2}\frac{(1+10s)^2(1+0.1s)}{2} = 5\frac{(1+10s)}{(1+0.1s)}$$

Which is the difference between the two solutions?
The difference can be found in terms of the control effort, comparing the control sensitivity functions in Figs. 4 and 5.

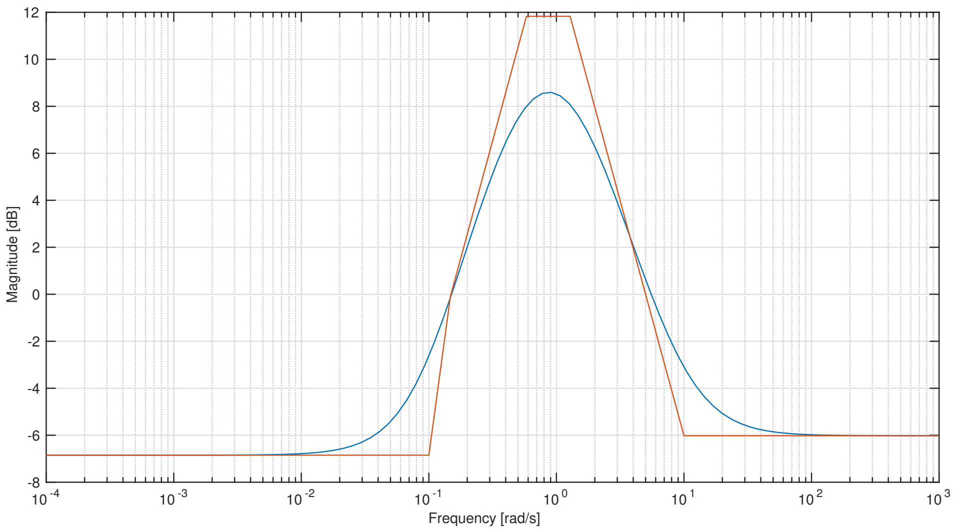


Figure 4: $Q_a(s)$ magnitude Bode diagram, first solution.

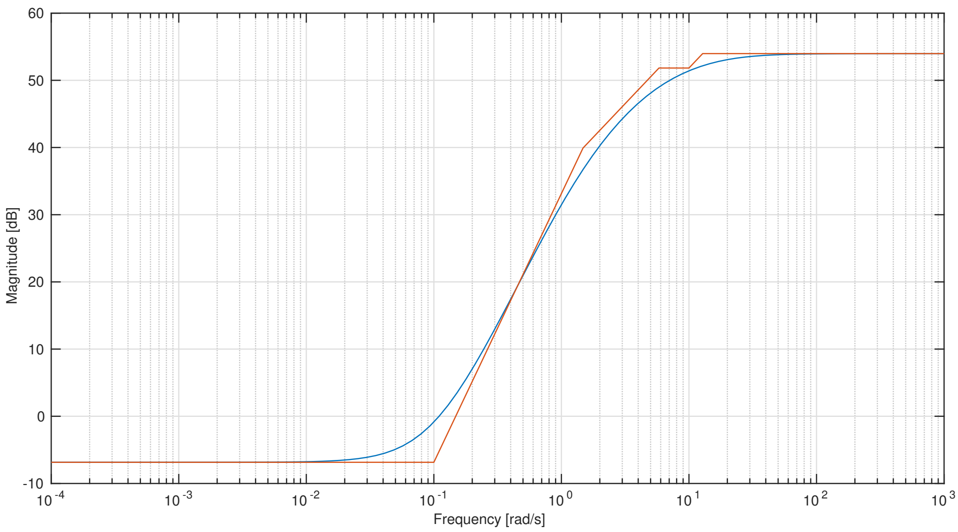
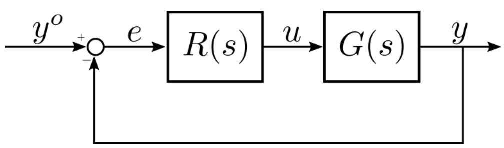


Figure 5: $Q_b(s)$ magnitude Bode diagram, second solution.

Exercise 2

Consider the following control system



where $G(s) = \dfrac{1}{s(1+10s)(1+s)}$.

Compute the transfer function $R(s)$ of a controller in such a way that:

- $|e_\infty| = 0$ for $y^o(t) = Asca(t)$, where A is an arbitrary real constant;
- the phase margin φ_m is greater or equal to 70^o;
- the crossover frequency ω_c is greater or equal to $0.1\, rad/s$.

Solution

Steady-state design
We start from the steady-state design, assuming that once the design will be completed the closed-loop system will be asymptotically stable.
The steady-state error due to the reference is

$$e_\infty = \lim_{s\to 0} s \frac{1}{1+L(s)}\frac{A}{s} = \lim_{s\to 0} \frac{A}{1+\frac{\mu_r}{s^{g_r+1}}} = \lim_{s\to 0} A\frac{s^{g_r+1}}{s^{g_r+1}+\mu_r} = 0 \qquad g_r = 0 \quad \forall \mu_r$$

We complete the steady-state design assuming $\mu_r = 1$ and moving the selection of the controller gain to the transient design, thus

$$R_1(s) = 1$$

Transient design
We start the transient design with

$$L_1(s) = \frac{1}{s(1+10s)(1+s)}$$

Magnitude Bode diagram of $L_1(s)$ is shown in Fig. 6.

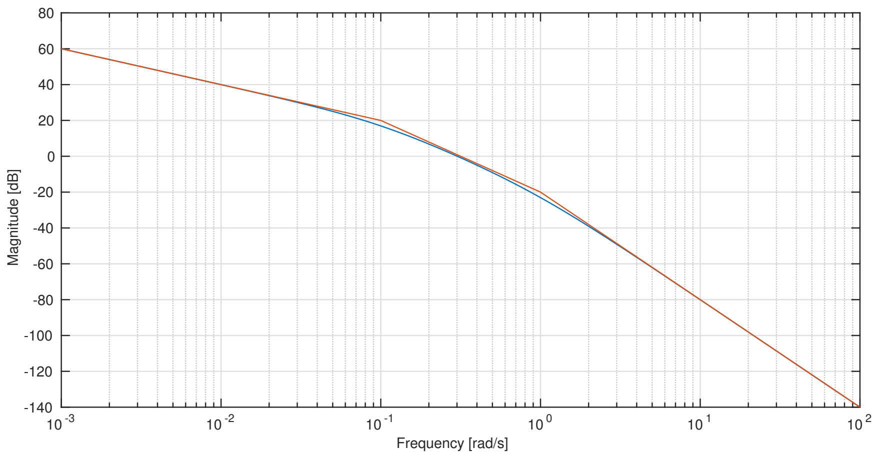


Figure 6: $L_1(s)$ magnitude Bode diagram.

Though the crossover frequency fulfils the requirements, having slope -2 at the crossover the phase margin cannot fulfil it, we thus need to revise the design.
The loop transfer function has to be reshaped in such a way that it has the same type of $L_1(s)$ (same slope at low frequency), the gain have still to be selected, and it has slope at least -3 at high frequency (regulator causality). As $G(s)$ is the transfer function of a minimum phase system we decide to cross the $0\,dB$-axis with slope -1 at $0.1\,rad/s$, the minimum crossover frequency that satisfies the requirement specifications. Doing in this way we obtain the loop transfer function depicted in Fig. 7.

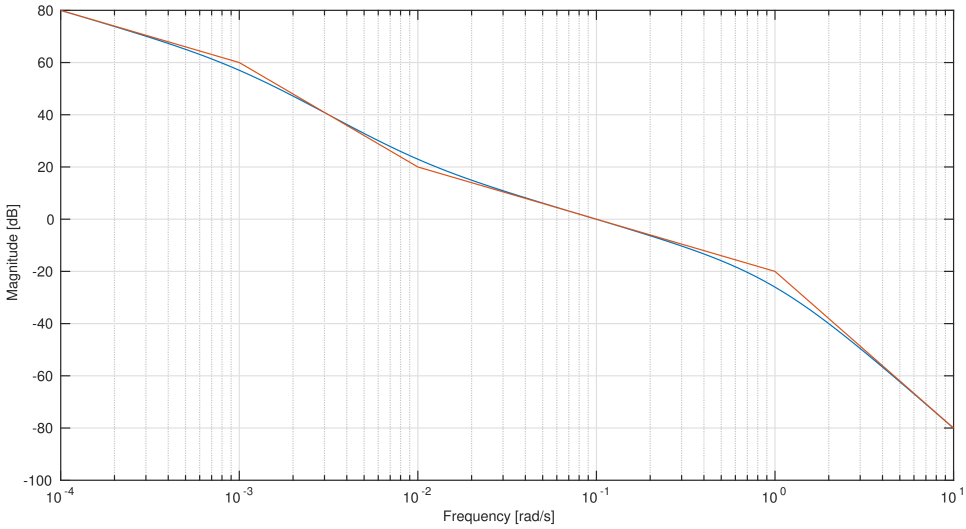


Figure 7: $L(s)$ magnitude Bode diagram.

The corresponding loop transfer function is

$$L(s) = \frac{1 + 100s}{s(1 + 1000s)(1 + s)^2}$$

which is characterised by

$$\omega_c = 0.1\, rad/s$$
$$\varphi_c = -90^o + \arctan(100 \cdot 0.1) - \arctan(1000 \cdot 0.1) - 2\arctan(0.1) \approx -106.5^o$$

and finally

$$\varphi_m = 180^o - |\varphi_c| \approx 73.5^o$$

The requirements on crossover frequency and phase margin are both fulfilled.
The corresponding regulator is

$$R(s) = \frac{L(s)}{G(s)} = \frac{1 + 100s}{s(1 + 1000s)(1 + s)^2} \frac{s(1 + 10s)(1 + s)}{1} = \frac{(1 + 100s)(1 + 10s)}{(1 + 1000s)(1 + s)}$$

As we have only to guarantee that $L(s)$ has the same slope of $L_1(s)$ at low frequency, a simpler solution would be

$$L(s) = \frac{0.1}{s(1 + s)^2}$$

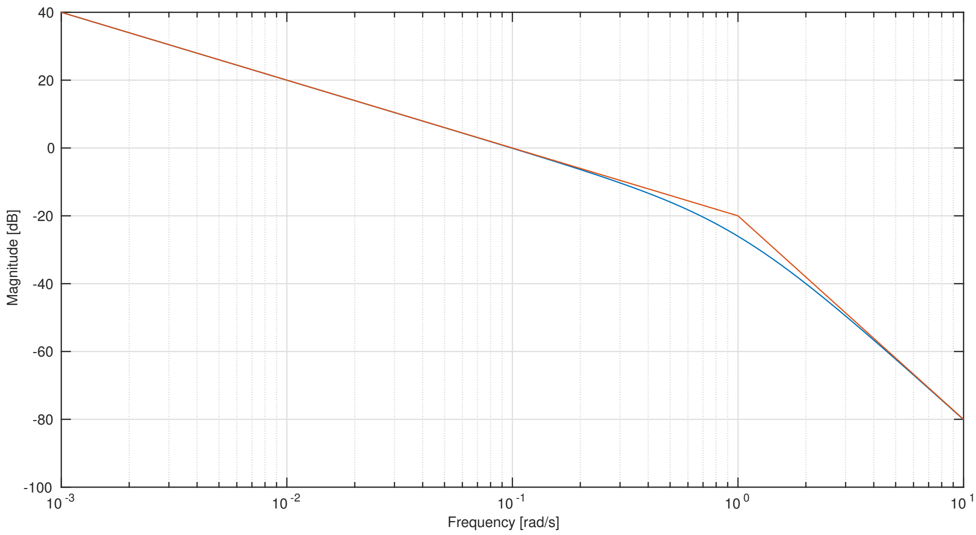


Figure 8: $L(s)$ magnitude Bode diagram.

This loop transfer function is characterised by (Fig. 8)

$$\omega_c = 0.1\,rad/s$$
$$\varphi_c = -90^o - 2\arctan(0.1) \approx -101.4^o$$

and finally

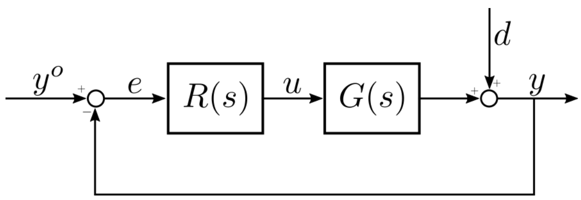
$$\varphi_m = 180^o - |\varphi_c| \approx 78.6^o$$

The requirements on crossover frequency and phase margin are both fulfilled.
The corresponding regulator is

$$R(s) = \frac{L(s)}{G(s)} = \frac{0.1}{s(1+s)^2}\frac{s(1+10s)(1+s)}{1} = 0.1\frac{1+10s}{1+s}$$

Exercise 3

Consider the following control system



where $G(s) = \frac{10}{s}\frac{1+0.1s}{1+s}$.
Compute the transfer function $R(s)$ of a controller in such a way that:

- $|e_\infty| \leq 0.1$ for $y^o(t) = \text{ram}(t)$ and $d(t) = 0$;
- a disturbance $d(t) = A\sin(\omega t)$, where A is an arbitrary constant and $\omega \leq 0.1\,rad/s$, is attenuated on the output of 1000 times;
- the phase margin φ_m is greater or equal to 80^o;
- the crossover frequency ω_c is greater or equal to $10\,rad/s$.

Solution

Steady-state design

We start from the steady-state design, assuming that once the design will be completed the closed-loop system will be asymptotically stable.

The steady-state error due to the reference is

$$e_\infty = \lim_{s\to 0} s \frac{1}{1+L(s)} \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{1+\frac{10\mu_r}{s^{g_r+1}}} \frac{1}{s} = \lim_{s\to 0} \frac{s^{g_r}}{s^{g_r+1}+10\mu_r} = \frac{1}{10\mu_r} \qquad g_r = 0$$

Enforcing the steady-state requirements we obtain

$$\frac{1}{10\mu_r} \le 0.1 \qquad \Rightarrow \qquad \mu_r \ge 1$$

We complete the steady-state design assuming $\mu_r = 1$, thus

$$R_1(s) = 1$$

We should now consider the effect of the sinusoidal disturbance on the controlled variable, the amplitude of the disturbance on the output is

$$\left| \frac{1}{1+L(j\omega)} \right|_{\omega \le 0.1} A$$

In order to ensure the required attenuation we need to fulfil, in the transient design, the following constraint

$$\left| \frac{1}{1+L(j\omega)} \right|_{\omega \le 0.1} \le \frac{1}{1000} \qquad \Rightarrow \qquad |L(j\omega)|_{\omega \le 0.1} \ge 60\, dB$$

Transient design

We start the transient design with

$$L_1(s) = \frac{10}{s} \frac{1+0.1s}{1+s}$$

Magnitude Bode diagram of $L_1(s)$ is shown in Fig. 9, where we can see that the first trial does not satisfy the requirement specifications.

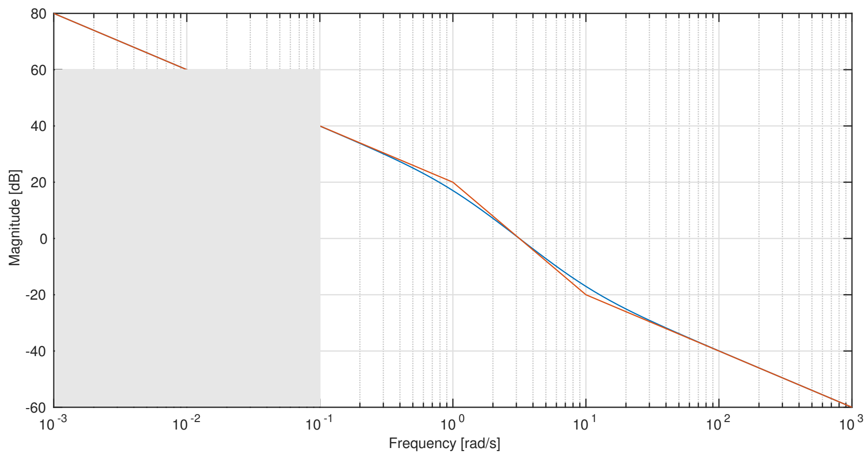


Figure 9: $L_1(s)$ magnitude Bode diagram.

The loop transfer function has to be reshaped in such a way that it has the same gain and type of $L_1(s)$ (same low frequency behaviour), it has slope at least -1 at high frequency (regulator causality), and it does not enter the forbidden region. As $G(s)$ is the transfer function of a minimum phase system we decide to cross the $0\,dB$-axis with slope -1 at $10\,rad/s$, the minimum crossover frequency that satisfies the requirement specifications. Doing in this way we obtain the loop transfer function depicted in Fig. 10.

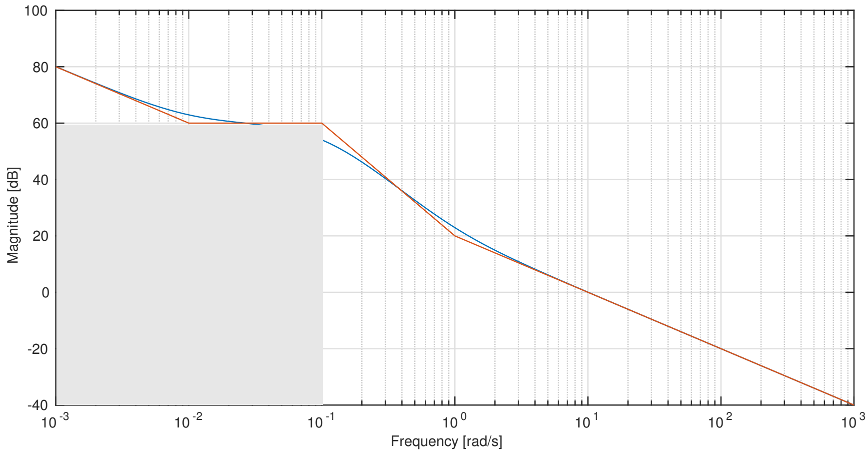


Figure 10: $L(s)$ magnitude Bode diagram.

The corresponding loop transfer function is

$$L(s) = \frac{10}{s}\frac{(1+100s)(1+s)}{(1+10s)^2}$$

which is characterised by

$$\omega_c = 10\, rad/s$$
$$\varphi_c = -90^o + \arctan(100 \cdot 10) + \arctan(10) - 2\arctan(10 \cdot 10) \approx -94.6^o$$

and finally

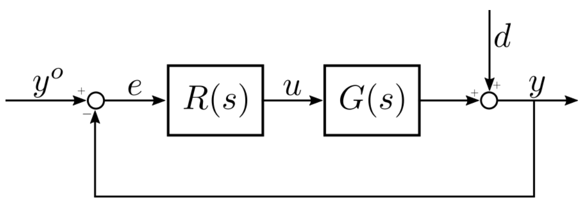
$$\varphi_m = 180^o - |\varphi_c| \approx 85.4^o$$

The requirements on crossover frequency, phase margin and disturbance attenuation are all fulfilled. The corresponding regulator is

$$R(s) = \frac{L(s)}{G(s)} = \frac{10}{s}\frac{(1+100s)(1+s)}{(1+10s)^2}\frac{s}{10}\frac{1+s}{1+0.1s} = \frac{(1+100s)(1+s)^2}{(1+0.1s)(1+10s)^2}$$

Exercise 4

Consider the following control system



where $G(s) = 10\dfrac{1-0.1s}{(1+10s)(1+0.01s)}$.
Compute the transfer function $R(s)$ of a controller in such a way that:

- $|e_\infty| \leq 0.02$ for $y^o(t) = 5\text{sca}(t)$ and $d(t) = 10\text{sca}(t)$;
- the phase margin φ_m is greater or equal to 75^o;
- the crossover frequency ω_c is greater or equal to $1\, rad/s$.

Solution

Steady-state design

We start from the steady-state design, assuming that once the design will be completed the closed-loop system will be asymptotically stable.

Using the superimposition principle we can decompose the steady-state error into two contributions, one due to the reference

$$e_{\infty_{y^o}} = \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{5}{s} = \lim_{s\to 0}\frac{5}{1+\frac{10\mu_r}{s^{g_r}}} = \lim_{s\to 0}\frac{5s^{g_r}}{s^{g_r}+10\mu_r} = \frac{5}{1+10\mu_r} \qquad g_r = 0$$

and another one due to the disturbance

$$e_{\infty_d} = \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{10}{s} = \lim_{s\to 0}\frac{10}{1+\frac{10\mu_r}{s^{g_r}}} = \lim_{s\to 0}\frac{10s^{g_r}}{s^{g_r}+10\mu_r} = \frac{10}{1+10\mu_r} \qquad g_r = 0$$

Enforcing the steady-state requirements we obtain

$$|e_\infty| = |e_{\infty_{y^o}} + e_{\infty_d}| \leq |e_{\infty_{y^o}}| + |e_{\infty_d}| = \frac{5}{1+10\mu_r} + \frac{10}{1+10\mu_r} = \frac{15}{1+10\mu_r} \leq 0.02 \qquad \Rightarrow \qquad \mu_r \geq 75$$

We complete the steady-state design assuming $\mu_r = 100$, thus

$$R_1(s) = 100$$

Transient design

We start the transient design with

$$L_1(s) = 1000\frac{1-0.1s}{(1+10s)(1+0.01s)}$$

Magnitude Bode diagram of $L_1(s)$ is shown in Fig. 11.

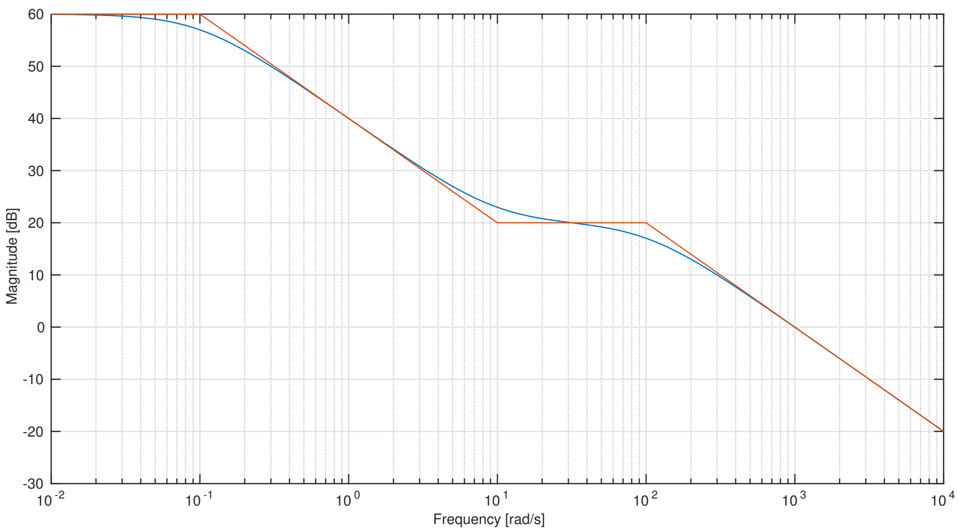


Figure 11: $L_1(s)$ magnitude Bode diagram.

Though the crossover frequency fulfils the requirements, having slope -2 at the crossover the phase margin cannot fulfil it, we thus need to revise the design.

The loop transfer function has to be reshaped in such a way that it has the same gain and type of $L_1(s)$ (same low frequency behaviour), it has slope at least -1 at high frequency (regulator causality), and crosses the $0\,dB$-axis before the frequency of the zero in the right half plane, that cannot be cancel out, in order to limit as much as possible its negative contribution to the phase. We decide to cross the $0\,dB$-axis with slope -1 at $1\,rad/s$, the minimum crossover frequency that satisfies the requirement specifications, one decade before the frequency of the zero. Doing in this way we obtain the loop transfer function depicted in Fig. 12.

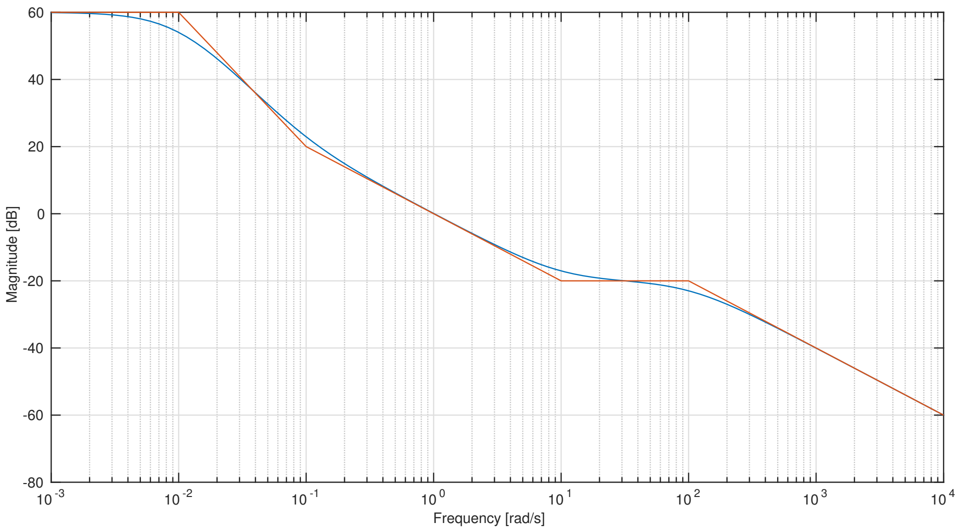


Figure 12: $L(s)$ magnitude Bode diagram.

The corresponding loop transfer function is

$$L(s) = 1000\frac{(1+10s)(1-0.1s)}{(1+100s)^2(1+0.01s)}$$

which is characterised by

$$\omega_c = 1\,rad/s$$
$$\varphi_c = \arctan(10) - \arctan(0.1) - 2\arctan(100) - \arctan(0.01) \approx -100.8^o$$

and finally

$$\varphi_m = 180^o - |\varphi_c| \approx 79.2^o$$

The requirements on crossover frequency and phase margin are both fulfilled.
The corresponding regulator is

$$R(s) = \frac{L(s)}{G(s)} = 1000\frac{(1+10s)(1-0.1s)}{(1+100s)^2(1+0.01s)}\frac{(1+10s)(1+0.01s)}{10(1-0.1s)} = 100\frac{(1+10s)^2}{(1+100s)^2}$$

Another solution that gives rise to a simpler regulator would be

$$L(s) = 1000\frac{1-0.1s}{(1+1000s)(1+0.01s)}$$

which is characterised by (Fig. 13)

$$\omega_c = 1\,rad/s$$
$$\varphi_c = -\arctan(0.1) - \arctan(1000) - \arctan(0.01) \approx -96.3^o$$

and finally

$$\varphi_m = 180^o - |\varphi_c| \approx 83.7^o$$

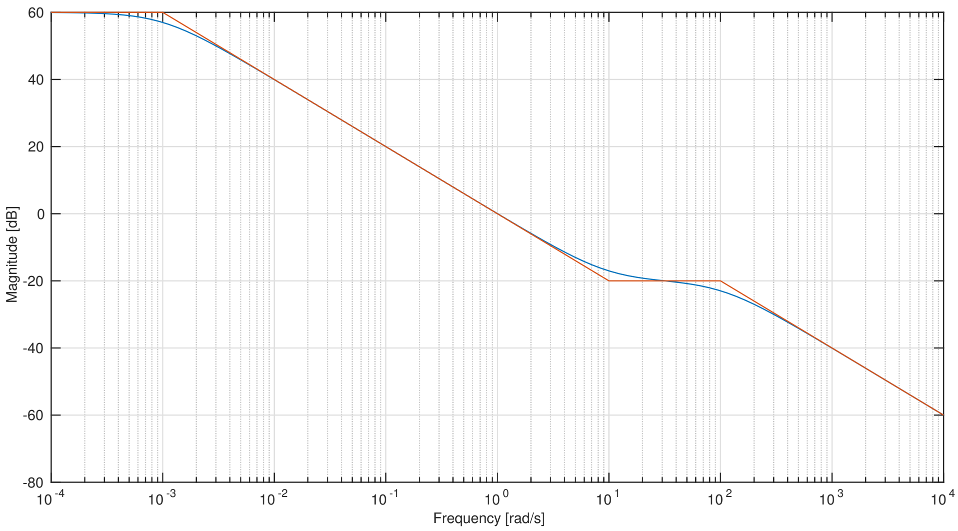


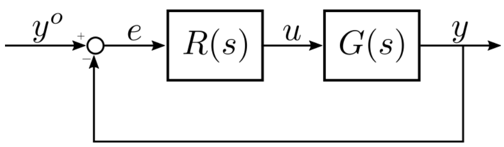
Figure 13: $L(s)$ magnitude Bode diagram.

The requirements on crossover frequency and phase margin are both fulfilled.
The corresponding regulator is

$$R(s) = \frac{L(s)}{G(s)} = 1000\frac{1-0.1s}{(1+1000s)(1+0.01s)}\frac{(1+10s)(1+0.01s)}{10(1-0.1s)} = 100\frac{1+10s}{1+1000s}$$

Exercise 5

Consider the following control system



where $G(s) = 10\frac{e^{-5s}}{(1+10s)(1+0.1s)}$.
Compute the transfer function $R(s)$ of a controller in such a way that:

- $|e_\infty| = 0$ for $y^o(t) = 10\text{sca}(t)$;
- the phase margin φ_m is greater or equal to 65^o;
- the crossover frequency is roughly maximized.

Solution

Steady-state design
We start from the steady-state design, assuming that once the design will be completed the closed-loop system will be asymptotically stable.
The steady-state error due to the reference is

$$e_\infty = \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{10}{s} = \lim_{s\to 0}\frac{10}{1+\frac{10\mu_r}{s^{g_r}}} = \lim_{s\to 0} 10\frac{s^{g_r}}{s^{g_r}+10\mu_r} = 0 \qquad g_r = 1 \quad \forall \mu_r$$

We complete the steady-state design assuming $\mu_r = 1$ and moving the selection of the controller gain to the transient design, thus

$$R_1(s) = \frac{1}{s}$$

Transient design

We start the transient design with

$$L_1(s) = 10\frac{e^{-5s}}{s(1+10s)(1+0.1s)}$$

Magnitude Bode diagram of $L_1(s)$ is shown in Fig. 14.

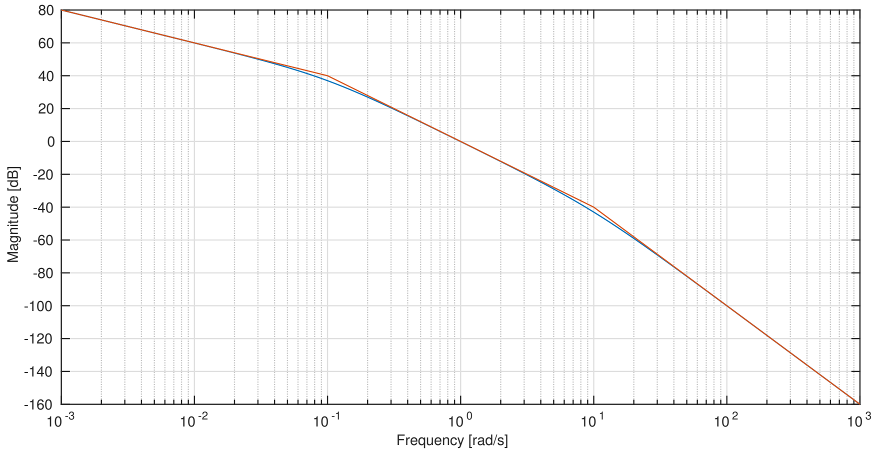


Figure 14: $L_1(s)$ magnitude Bode diagram.

Having slope -2 at the crossover and due to the effect of the delay, the phase margin cannot fulfil the requirements, we thus need to revise the design.
The loop transfer function has to be reshaped in such a way that it has the same type of $L_1(s)$ (same slope at low frequency), the gain have still to be selected, and it has slope at least -3 at high frequency (regulator causality). Moreover, we must consider that the delay introduces a decrement in the phase that is proportional to the crossover frequency.
We can cancel out all the poles of $G(s)$, apart from the integrator, and introduce then a couple of high frequency poles to achieve slope -3 without affecting the phase margin, that will be approximately equal to

$$\varphi_m = 180^o - 90^o - 5\omega_c\frac{180^o}{\pi}$$

Enforcing the constraint $\varphi_m \geq 65^o$ we can roughly determine the maximum crossover frequency

$$\omega_c \leq \frac{25^o}{5}\frac{\pi}{180^o} \approx 0.087\,rad/s$$

we could thus select a crossover frequency equal to $0.06\,rad/s$.
Doing in this way we obtain the loop transfer function depicted in Fig. 15.

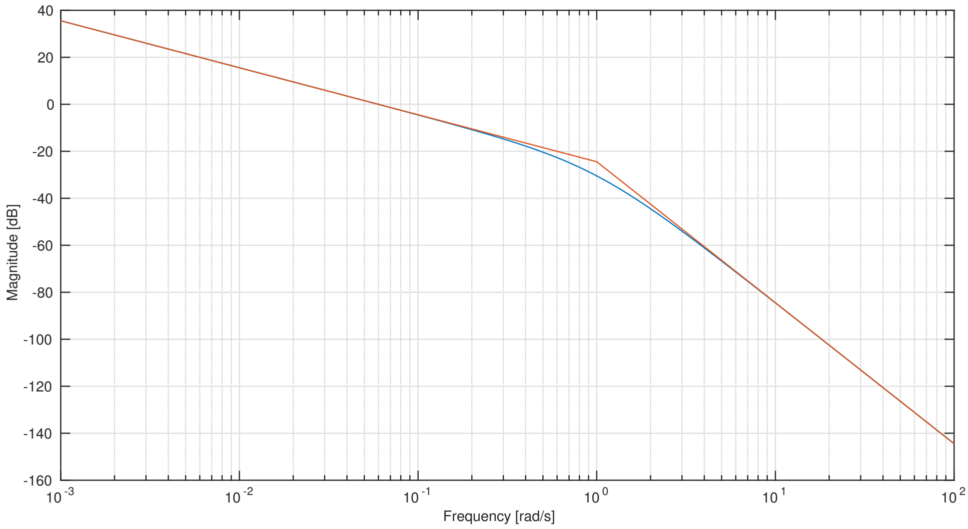


Figure 15: $L(s)$ magnitude Bode diagram.

The corresponding loop transfer function is

$$L(s) = \frac{0.06}{s(1+s)^2} e^{-5s}$$

which is characterised by

$$\omega_c = 0.06\, rad/s$$

$$\varphi_c = -90^o - 2\arctan(0.06) - 0.06 \cdot 5\frac{180}{\pi} \approx -114^o$$

and finally

$$\varphi_m = 180^o - |\varphi_c| \approx 66^o$$

The requirements on phase margin is fulfilled and the crossover frequency is roughly maximised. The corresponding regulator is

$$R(s) = \frac{L(s)}{G(s)} = 0.06\frac{e^{-5s}}{s(1+s)^2}\frac{(1+10s)(1+0.1s)}{10e^{-5s}} = 0.006\frac{(1+10s)(1+0.1s)}{s(1+s)^2}$$