POLITECNICO DI MILANO

Automatic Control

Systems theory overview (continuous time systems)

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Motivations

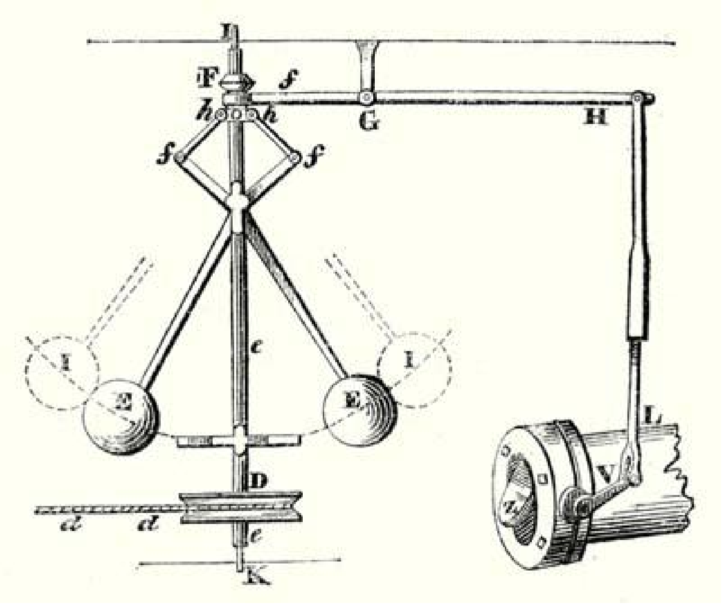
We will start introducing the fundamentals of systems theory.

This basic knowledge will represent the ground on which we will develop basic and advanced control theory.

The main topics we will face are:

- fundamentals of dynamical systems
- solutions and equilibrium points
- Lyapunov stability
- Linear and Time Invariant systems
- solutions and equilibrium points for LTI systems
- stability of LTI systems
- stability analysis of LTI systems
- stability of equilibria of nonlinear systems
- transfer function of a LTI system
- observability and controllability
- realization and canonical forms

The control problem (I)

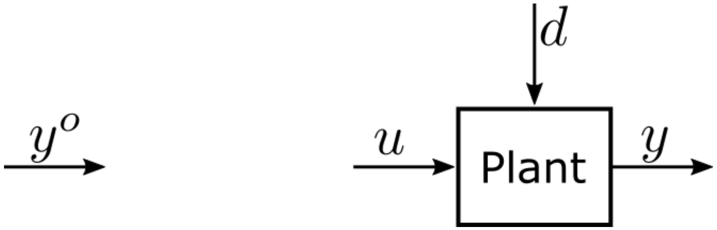


What is a control problem?

We have a control problem all the times we would like to make a machine behaving in a desired way.

The controller is the device that exerts the appropriate actions on the machine, in order to achieve the desired behavior.

The control law is the algorithm the controller uses to determine the action to be performed.



Let's try to formalize…

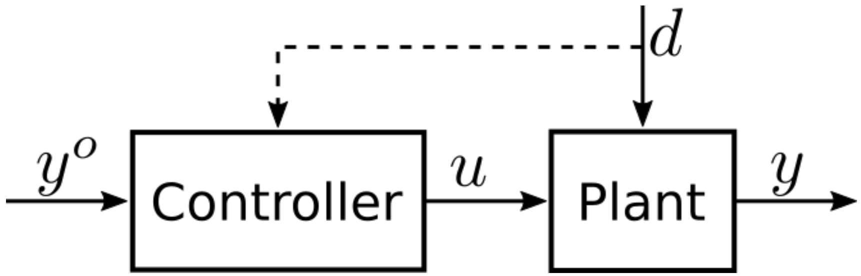
The controller determines, at every time instant, the value of the control u in such a way that the controlled variable y is as similar as possible to its reference y^o, for every "reasonable" behavior of the reference and the disturbance d.

The control problem (II)

How can a control system be designed?

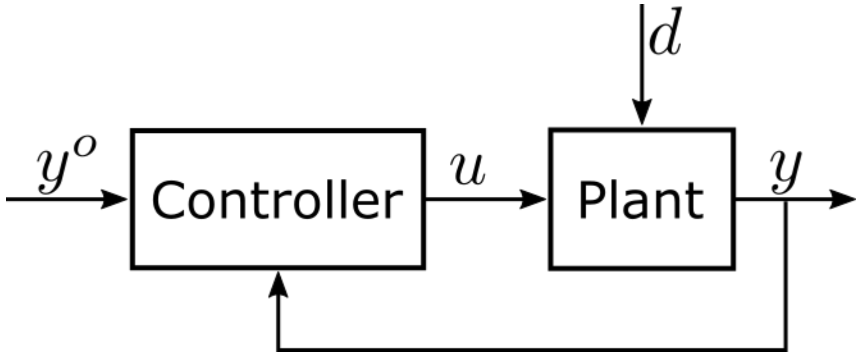
<u>Open-loop control</u> (feedforward)

No measured variable is used to compute the control variable, or measured variables that do not depend on u are used.



<u>Closed-loop control</u> (feedback)

Measurements are used to compute the control variable, whose values depend on the control variable itself.



The control problem (III)

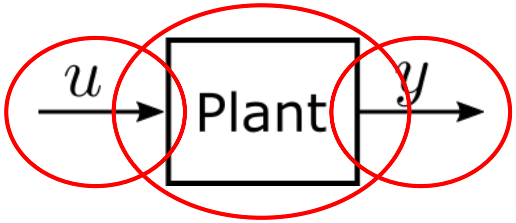
We will start studying how a controlled system can be represented and analyzed.

We will make reference to the tools offered by systems theory, that allow to study the properties of physical systems irrespective of their physical domain (mechanical, electrical, hydraulic, etc.).

We will focus on a dynamical description of the physical system as we are interested to describe its time evolution.

Dynamical systems (I)

A dynamical system is a mathematical tool to represent a physical system by way of a model



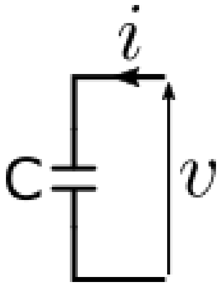
Input (u) and output (y) variables are the interfaces allowing the system to interact with the environment in which it operates.

Input variables u allow the environment to act on the system, affecting its behavior.

Output variables y are measurements we can use to monitor the behavior of the system.

The relationship between input and output variables established by the dynamical system is a "cause and effect" relationship, it is not a relationship involving flow of mass or energy.

Let's consider a very simple electrical system.



Input u: the current flowing in the capacitor

Output y: the voltage across the capacitor

The input-output relation describing the dynamical system is

$$C\dot{y}(t) = u(t)$$

Solving the differential equation we obtain the expression of the output

$$y(t) = y(t_0) + \frac{1}{C}\int_{t_0}^{t} u(\tau)\mathrm{d}\tau$$

Conclusions:

- even in a very simple system the input-output relation is differential (not just an algebraic relation)
- to compute the output at time t we need the initial value of the output and the values of the input for $t \geq t_0$

Dynamical systems (III)

We call order n of the dynamical system the minimum number of initial conditions we need, in order to compute the system output given the input values from the initial time.

We call state of the dynamical system the smallest set of linearly independent variables $(x_1, x_2, \dots, x_n)$ such that the values of the members of this set at time t_0, along with a known input function, completely determine the values of the state itself (and of the output) for all $t \geq t_0$.

Let's consider again the generic dynamical system.

From now on the input, output and state variables will be vectors of dimension m, p, and n, respectively. They will be represented, in vector notation, as follows

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

Let's consider again the input-output relation of the electrical system, we choose the differential one

$$C\dot{y}(t) = u(t)$$

We can generalize this relation, giving rise to the general vector equation of a dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

State equations

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t)$$

Output equations

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$, and

$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t) \\ f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m, t) \end{bmatrix}$$

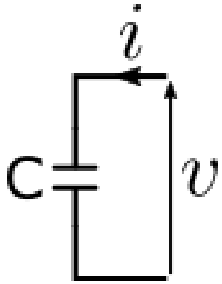
Dynamical systems (V)

$$\mathbf{g}(\mathbf{x}(t),\mathbf{u}(t),t) = \begin{bmatrix} g_1(x_1,x_2,\ldots,x_n,u_1,u_2,\ldots,u_m,t) \\ g_2(x_1,x_2,\ldots,x_n,u_1,u_2,\ldots,u_m,t) \\ \vdots \\ g_p(x_1,x_2,\ldots,x_n,u_1,u_2,\ldots,u_m,t) \end{bmatrix}$$

We will now introduce some examples of dynamical systems related to different physical domains (mechanical, electrical, hydraulic).

Dynamical systems – Examples (I)

$$i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t}$$



Capacitor

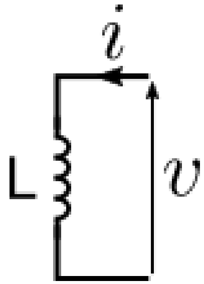
Input $u = i$

Output $y = v$

State $x_1 = v$

$$\dot{x}_1(t) = \frac{1}{C}u(t)$$
$$y(t) = x_1(t)$$

$$v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t}$$



Inductor

Input $u = v$

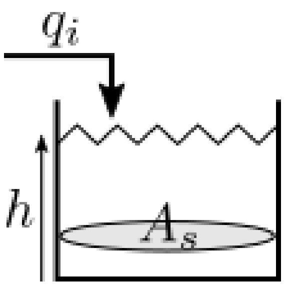
Output $y = i$

State $x_1 = i$

$$\dot{x}_1(t) = \frac{1}{L}u(t)$$
$$y(t) = x_1(t)$$

Dynamical systems – Examples (II)

$$q_i(t) = A_s \frac{\mathrm{d}h(t)}{\mathrm{d}t}$$



Tank

Input $u = q_i$

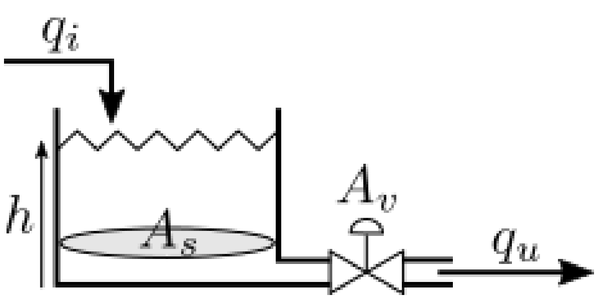
Output $y = h$

State $x_1 = h$

$$\dot{x}_1(t) = \frac{1}{A_s} u(t)$$

$$y(t) = x_1(t)$$

$$q_i(t) = A_s \frac{\mathrm{d}h(t)}{\mathrm{d}t} + kA_v\sqrt{h(t)}$$



Tank with valve

Input $u = q_i$

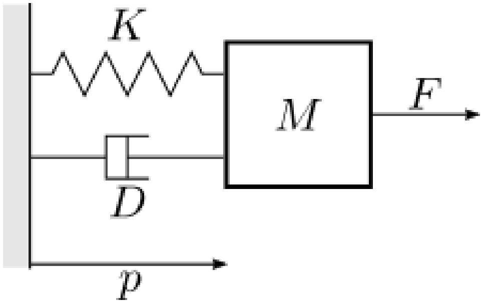
Output $y = h$

State $x_1 = h$

$$\dot{x}_1(t) = -k\frac{A_v}{A_s}\sqrt{x_1(t)} + \frac{1}{A_s} u(t)$$

$$y(t) = x_1(t)$$

$$F(t) = M\dot{v}(t) + Dv(t) + Kp(t)$$

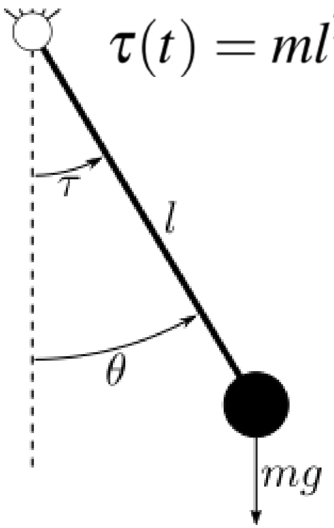


Mass-spring-damper

Input	$u = F$
Output	$y = p$
State	$x_1 = p$
	$x_2 = v$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{K}{M}x_1(t) - \frac{D}{M}x_2(t) + \frac{1}{M}u(t)$$
$$y(t) = x_1(t)$$

$$\tau(t) = ml^2\dot{\omega}(t) + mgl\sin(\theta(t))$$



Simple pendulum

Input	$u = \tau$
Output	$y = \theta$
State	$x_1 = \theta$
	$x_2 = \omega$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{g}{l}\sin(x_1(t)) + \frac{1}{ml^2}u(t)$$
$$y(t) = x_1(t)$$

There are many possible ways to classify dynamical systems, let's see the most important ones.

SISO (Single Input Single Output) – if $m = p = 1$

MIMO (Multi Input Multi Output) – if $m > 1$ or $p > 1$

Strictly proper – if function g does not depend on the input (i.e., $g(\mathbf{x}(t), t)$)

Proper – if function g depends on the input

Linear – if all the functions f_i and g_i are linear with respect to state and input variables

Non linear – if there is at least one function f_i or g_i that is not linear with respect to at least a state or an input variable

Time invariant – if functions f and g do not depend on time t (i.e., $f(\mathbf{x}(t), \mathbf{u}(t))$ and $g(\mathbf{x}(t), \mathbf{u}(t))$)

Time varying – if there is at least one function f_i or g_i that depends on time t

Solutions and equilibrium points (I)

Given a dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad \mathbf{x} \in \mathbb{R}^n \quad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad \mathbf{y} \in \mathbb{R}^p$$

an initial condition at time t_0

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

and an input function for all $t \geq t_0$

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) \quad t \geq t_0$$

we call

- <u>state trajectory</u>

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{f}(\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t), t) \qquad \bar{\mathbf{x}}(t) \in \mathbb{R}^n$$

$$\bar{\mathbf{x}}(t_0) = \mathbf{x}_0$$

- <u>output trajectory</u>

$$\bar{\mathbf{y}}(t) = \mathbf{g}(\bar{\mathbf{x}}(t), \bar{\mathbf{u}}(t), t) \qquad \bar{\mathbf{y}}(t) \in \mathbb{R}^p$$

Solutions and equilibrium points (II)

A constant trajectory, generated by a constant input function, is called equilibrium.

Given a time-invariant dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \qquad \mathbf{x} \in \mathbb{R}^n \quad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \qquad \mathbf{y} \in \mathbb{R}^p$$

and a constant input function for all $t \geq t_0$

$$\mathbf{u}(t) = \bar{\mathbf{u}} \quad t \geq t_0$$

we call

- state equilibrium

$$\mathbf{x}(t) = \bar{\mathbf{x}} \quad \forall t$$

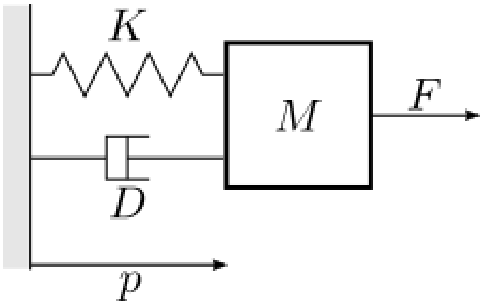
- output equilibrium

$$\mathbf{y}(t) = \bar{\mathbf{y}} \quad \forall t$$

The equilibria are solutions of the following equations

$$\mathbf{0} = \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$$

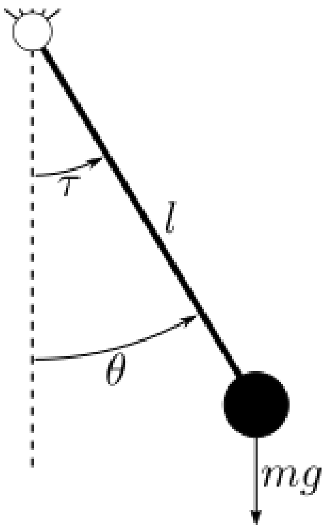
$$\bar{\mathbf{y}} = \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$$



Mass-spring-damper

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{K}{M}x_1(t) - \frac{D}{M}x_2(t) + \frac{1}{M}u(t)$$
$$y(t) = x_1(t)$$

$$\begin{aligned} 0 &= \bar{x}_2 \\ 0 &= -K\bar{x}_1 - D\bar{x}_2 + \bar{u} \end{aligned} \quad \Rightarrow \quad \bar{x}_1 = \frac{\bar{u}}{K} \quad \bar{x}_2 = 0$$



Simple pendulum

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{g}{l}\sin(x_1(t)) + \frac{1}{ml^2}u(t)$$
$$y(t) = x_1(t)$$

$$\begin{aligned} 0 &= \bar{x}_2 \\ 0 &= -mgl\sin(\bar{x}_1) + \bar{u} \end{aligned} \quad \overset{\bar{u}=0}{\Rightarrow} \quad \begin{cases} \bar{x}_1 = 0 \\ \bar{x}_2 = 0 \end{cases} \quad \begin{cases} \bar{x}_1 = \pi \\ \bar{x}_2 = 0 \end{cases}$$

Stability theory studies how trajectories of a dynamical system change under small perturbations of initial conditions.

We will make reference to Lyapunov stability theory and concentrate our analysis on time invariant systems.

Given a general time invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$$

an input function

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) \quad t > 0$$

and two initial conditions

- $\mathbf{x}_{0_n}$ the nominal initial condition
- $\mathbf{x}_{0_p}$ the perturbed initial condition

two trajectories are generated

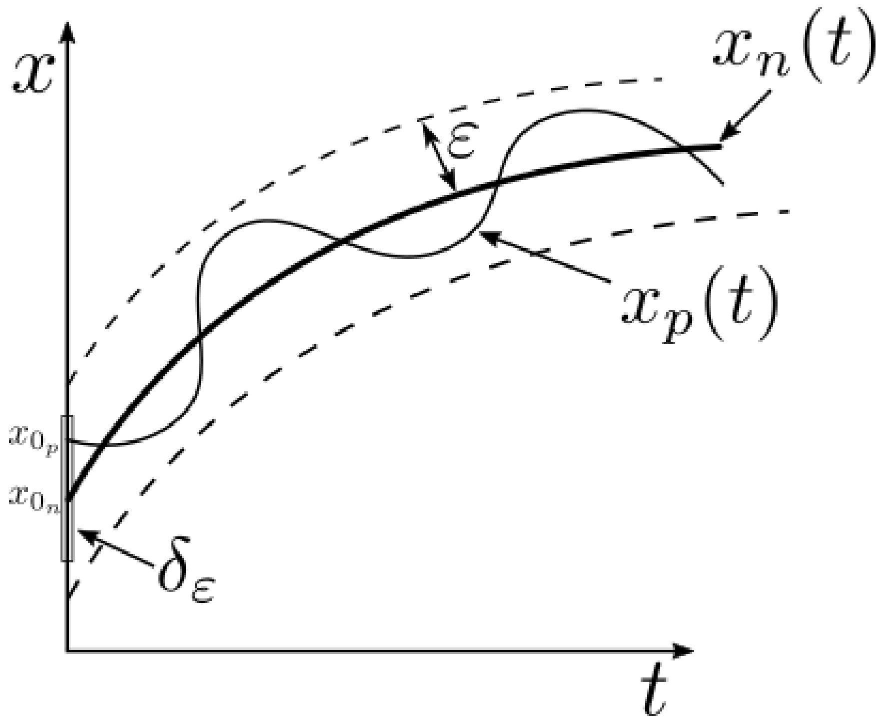
- $\mathbf{x}_n(t)$, nominal trajectory, generated by $\mathbf{x}_{0_n}$ and $\bar{\mathbf{u}}(t)$
- $\mathbf{x}_p(t)$, perturbed trajectory, generated by $\mathbf{x}_{0_p}$ and $\bar{\mathbf{u}}(t)$

Lyapunov stability (II)

$\mathbf{x}_n(t)$ is stable if

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \text{s.t.}$$

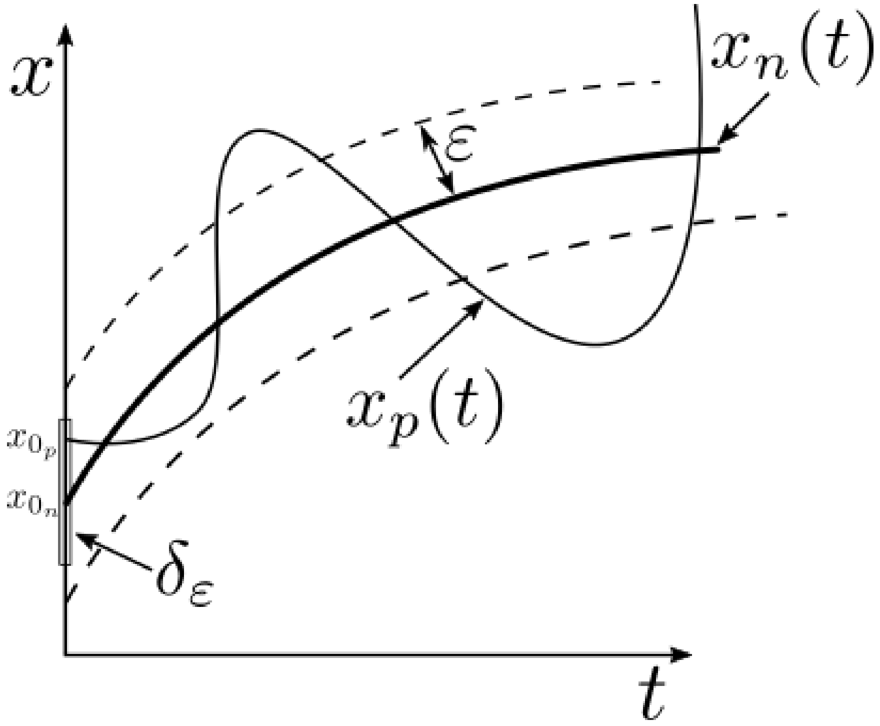
$$\forall \mathbf{x}_{0_p} : \|\mathbf{x}_{0_p} - \mathbf{x}_{0_n}\| \leq \delta_\varepsilon \Rightarrow \|\mathbf{x}_p(t) - \mathbf{x}_n(t)\| \leq \varepsilon \quad \forall t \geq 0$$



The trajectory generated by perturbing the initial condition (perturbed trajectory) remains closed to the nominal one.

Lyapunov stability (III)

$\mathbf{x}_n(t)$ is unstable if it is not stable

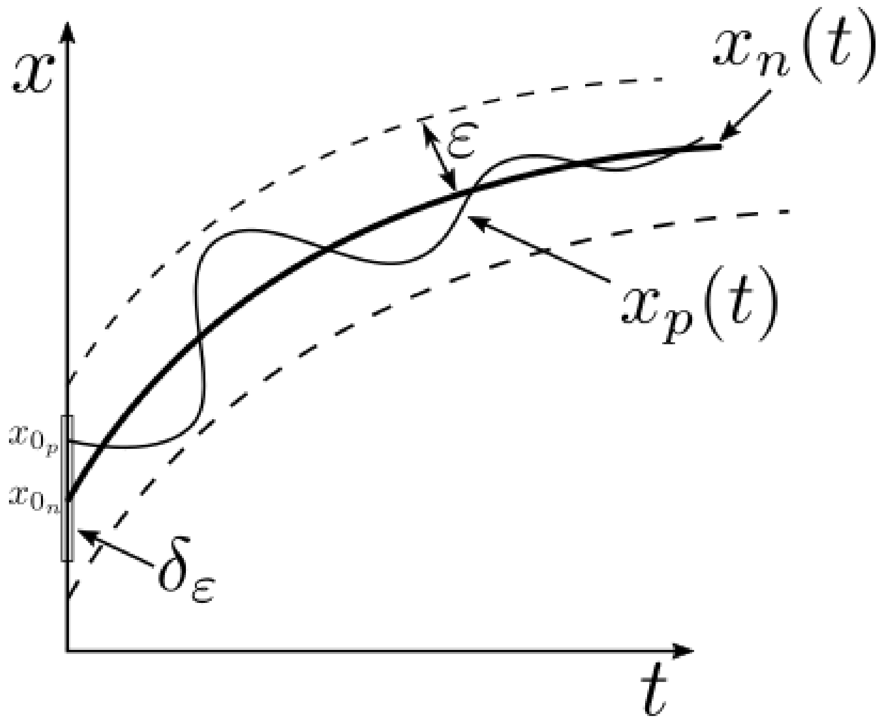


The trajectory generated by perturbing the initial condition (perturbed trajectory) get away from the nominal one.

Lyapunov stability (IV)

$\mathbf{x}_n(t)$ is <u>asymptotically stable</u> if it is stable and

$$\lim_{t \to \infty} \|\mathbf{x}_p(t) - \mathbf{x}_n(t)\| = 0 \quad \forall \mathbf{x}_{0_p} : \|\mathbf{x}_{0_p} - \mathbf{x}_{0_n}\| \leq \delta_\varepsilon$$



The trajectory generated by perturbing the initial condition (perturbed trajectory) remains closed to the nominal one and asymptotically tends to it.

Lyapunov stability (V)

The previous definitions (stable, unstable, asymptotically stable) hold for equilibria as well, as they are constant trajectories.

In LTI systems all states and output equations are linear with respect to all state and input variables.

Functions $\mathbf{f}$ and $\mathbf{g}$ can be thus represented as linear combinations

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_{11}u_1(t) + b_{12}u_2(t) + \cdots + b_{1m}u_m(t) \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_{21}u_1(t) + b_{22}u_2(t) + \cdots + b_{2m}u_m(t) \\ \vdots \\ \dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_{n1}u_1(t) + b_{n2}u_2(t) + \cdots + b_{nm}u_m(t) \end{cases}$$

$$\begin{cases} y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) + \cdots + c_{1n}x_n(t) + d_{11}u_1(t) + d_{12}u_2(t) + \cdots + d_{1m}u_m(t) \\ y_2(t) = c_{21}x_1(t) + c_{22}x_2(t) + \cdots + c_{2n}x_n(t) + d_{21}u_1(t) + d_{22}u_2(t) + \cdots + d_{2m}u_m(t) \\ \vdots \\ y_p(t) = c_{n1}x_1(t) + c_{n2}x_2(t) + \cdots + c_{nn}x_n(t) + d_{n1}u_1(t) + d_{n2}u_2(t) + \cdots + d_{nm}u_m(t) \end{cases}$$

and expressed in vector form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

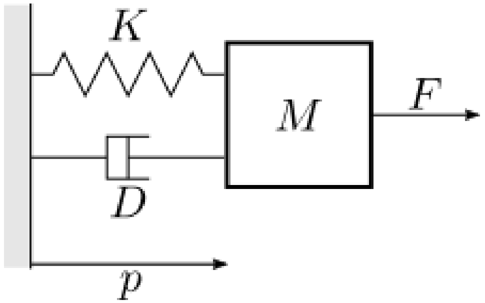
where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$, and

Linear Time Invariant systems (II)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \qquad D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}$$

Linear Time Invariant systems – Example



Mass-spring-damper

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{K}{M}x_1(t) - \frac{D}{M}x_2(t) + \frac{1}{M}u(t)$$

$$y(t) = x_1(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D}{M} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

Linear Time Invariant systems (III)

A state variable representation of a LTI system is not unique, there are infinitely many representations that are equivalent from the input-output point of view.

Consider the state space representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

we introduce the following change of variables

$$\hat{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t) \qquad \det(\mathbf{T}) \neq 0$$

and, thanks to the non singularity of matrix T

$$\mathbf{x}(t) = \mathbf{T}^{-1}\hat{\mathbf{x}}(t)$$

Differentiating the equation that defines the change of variables we obtain

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{T}\dot{\mathbf{x}}(t) = \mathbf{T}\mathbf{A}\mathbf{x}(t) + \mathbf{T}\mathbf{B}\mathbf{u}(t)$$

$$= \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\hat{\mathbf{x}}(t) + \mathbf{T}\mathbf{B}\mathbf{u}(t)$$

and substituting the change of variables into the output equation

Linear Time Invariant systems (IV)

$$\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) = \mathbf{C}\mathbf{T}^{-1}\hat{\mathbf{x}}(t) + \mathbf{D}\mathbf{u}(t)$$

Summarizing, the equations of the system in the new variables are

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t)$$

where

$$\hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \quad \hat{\mathbf{B}} = \mathbf{T}\mathbf{B} \quad \hat{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1} \quad \hat{\mathbf{D}} = \mathbf{D}$$

All the properties of a LTI system that are invariant with respect to a change of variables are called structural properties.

Solutions and equilibrium points (I)

Given a LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

and a constant input function for all $t > 0$

$$\mathbf{u}(t) = \bar{\mathbf{u}} \quad t > 0$$

state equilibria are solutions of the following equation

$$\mathbf{A}\bar{\mathbf{x}} + \mathbf{B}\bar{\mathbf{u}} = 0$$

If matrix $\mathbf{A}$ is non singular, there exists a unique state equilibrium given by

$$\bar{\mathbf{x}} = -\mathbf{A}^{-1}\mathbf{B}\bar{\mathbf{u}}$$

and the output equilibrium is

$$\bar{\mathbf{y}} = \boxed{\left(-\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}\right)}\bar{\mathbf{u}}$$

Static gain

Solutions and equilibrium points (II)

Given a LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

an initial condition

$$\mathbf{x}(0) = \mathbf{x}_0$$

and an input function for all $t \geq 0$

$$\mathbf{u} = \mathbf{u}(t) \quad t \geq 0$$

the state and output trajectories are given by

$$\mathbf{x}(t) = \boxed{\mathrm{e}^{\mathbf{A}t}\mathbf{x}_0} + \boxed{\int_0^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau}$$

$$\mathbf{y}(t) = \boxed{\mathbf{C}\mathrm{e}^{\mathbf{A}t}\mathbf{x}_0} + \boxed{\mathbf{C}\int_0^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau + \mathbf{D}\mathbf{u}(t)}$$

<u>Zero-input response</u>
Generated by the initial condition only

<u>Zero-state response</u>
Generated by the input only

Solutions and equilibrium points (III)

How can we compute the matrix exponential that appears in the system trajectories?

Given a matrix $\mathbf{A}$ and a scalar t the matrix exponential is defined as

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \dots$$

From the definition we can compute the derivative of the matrix exponential

$$\begin{aligned} \frac{d}{dt} e^{\mathbf{A}t} &= \mathbf{0} + \mathbf{A} + 2\frac{\mathbf{A}^2 t}{2!} + 3\frac{\mathbf{A}^3 t^2}{3!} + \dots \\ &= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \dots\right) \\ &= \mathbf{A}e^{\mathbf{A}t} \end{aligned}$$

Another interesting property of the matrix exponential…

Assume that $\mathbf{A}$ is a diagonalizable matrix

$$\exists \mathbf{T},\ \det(\mathbf{T}) \neq 0 \ : \ \hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_n\right)$$

Then

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \mathrm{e}^{\mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}t} = \mathbf{I} + \mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}t + \mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}\mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}\frac{t^2}{2!} + \ldots \\ &= \mathbf{T}^{-1}\left[\mathbf{I} + \hat{\mathbf{A}}t + \frac{\hat{\mathbf{A}}^2 t^2}{2!} + \ldots\right]\mathbf{T} \\ &= \mathbf{T}^{-1}\mathrm{e}^{\hat{\mathbf{A}}t}\mathbf{T} = \mathbf{T}^{-1}\mathrm{diag}\left(\mathrm{e}^{\lambda_1 t}, \ldots, \mathrm{e}^{\lambda_n t}\right)\mathbf{T} \end{aligned}$$

Stability of LTI systems (I)

Given a LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

an input function

$$\mathbf{u}(t) = \bar{\mathbf{u}}(t) \quad t > 0$$

and two initial conditions

- $\mathbf{x}_{0_n}$ the nominal initial condition
- $\mathbf{x}_{0_p}$ the perturbed initial condition

two trajectories are generated

- the nominal trajectory $\mathbf{x}_n(t)$, generated by $\mathbf{x}_{0_n}$ and $\bar{\mathbf{u}}(t)$
- the perturbed trajectory $\mathbf{x}_p(t)$, generated by $\mathbf{x}_{0_p}$ and $\bar{\mathbf{u}}(t)$

Recall that $\mathbf{x}_n(t)$ is <u>stable</u> if

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \text{s.t.}$$

$$\forall \mathbf{x}_{0_p} : \|\mathbf{x}_{0_p} - \mathbf{x}_{0_n}\| \leq \delta_\varepsilon \Rightarrow \|\mathbf{x}_p(t) - \mathbf{x}_n(t)\| \leq \varepsilon \quad \forall t \geq 0$$

Defining

$$\delta \mathbf{x}(t) = \mathbf{x}_p(t) - \mathbf{x}_n(t) \qquad \delta \mathbf{x}_0 = \mathbf{x}_{0_p} - \mathbf{x}_{0_n}$$

The stability definition becomes

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \text{s.t.}$$

$$\forall \mathbf{x}_{0_p} : \|\delta \mathbf{x}_0\| \leq \delta_\varepsilon \Rightarrow \|\delta \mathbf{x}(t)\| \leq \varepsilon \quad \forall t \geq 0$$

To asses the stability we have two compute the difference between the nominal and perturbed trajectories.

As both trajectories are solutions of the state differential equations we have

$$\dot{\mathbf{x}}_n(t) = \mathbf{A}\mathbf{x}_n(t) + \mathbf{B}\bar{\mathbf{u}}(t) \quad \mathbf{x}_n(0) = \mathbf{x}_{n_0}$$

$$\dot{\mathbf{x}}_p(t) = \mathbf{A}\mathbf{x}_p(t) + \mathbf{B}\bar{\mathbf{u}}(t) \quad \mathbf{x}_p(0) = \mathbf{x}_{p_0}$$

and doing the difference, side by side, between the two equations

$$\begin{array}{rclcrcl} \dot{\mathbf{x}}_n(t) & = & \mathbf{A}\mathbf{x}_n(t) + \mathbf{B}\bar{\mathbf{u}}(t) & & \mathbf{x}_n(0) & = & \mathbf{x}_{n_0} \\ \dot{\mathbf{x}}_p(t) & = & \mathbf{A}\mathbf{x}_p(t) + \mathbf{B}\bar{\mathbf{u}}(t) & & \mathbf{x}_p(0) & = & \mathbf{x}_{p_0} \\ \hline \delta \dot{\mathbf{x}}(t) & = & \mathbf{A}\delta \mathbf{x}(t) & & \delta \mathbf{x}(0) & = & \delta \mathbf{x}_0 \end{array}$$

Stability of LTI systems (III)

The stability of the nominal trajectory depends on the zero-input solution of the following autonomous LTI system

$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}\delta\mathbf{x}(t) \qquad \delta\mathbf{x}(0) = \delta\mathbf{x}_0$$

Let’s draw the first conclusions, in a LTI system:

- stability analysis depends on the zero-input solution of an autonomous system
- stability does not depend on the input function $\mathbf{u}(t)$
- stability depends only on the state matrix $\mathbf{A}$
- the result of stability analysis is independent of the chosen nominal trajectory
- the trajectories are all stable, all unstable or all asymptotically stable
- stability is a property of the system

Stability of LTI systems (IV)

The solution (zero-input trajectory) of the autonomous LTI system

$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}\delta\mathbf{x}(t) \quad \delta\mathbf{x}(0) = \delta\mathbf{x}_0$$

is

$$\delta\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\delta\mathbf{x}_0$$

The stability of the LTI system depends on these trajectories, and, interpreting the Lyapunov stability definition, we can say that the system is

- stable if all the zero-input trajectories are bounded
- asymptotically stable if all the zero-input trajectories are bounded and tend asymptotically to zero
- unstable if at least one of the zero-input trajectories is not bounded

Assuming that the state matrix $\mathbf{A}$ is diagonalizable, we can introduce a change of variables that decouples the trajectories and simplifies the computation of the matrix exponential

$$\exists\mathbf{T},\ \det(\mathbf{T}) \neq 0\ :\ \hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_n\right)$$

Stability of LTI systems (V)

The trajectories are given by

$$\delta\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\delta\mathbf{x}_0 = \mathrm{e}^{\mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}t}\delta\mathbf{x}_0$$

$$= \mathbf{T}^{-1}\mathrm{e}^{\hat{\mathbf{A}}t}\mathbf{T}\delta\mathbf{x}_0 = \mathbf{T}^{-1}\mathrm{diag}\left(\mathrm{e}^{\lambda_1 t},\ldots,\mathrm{e}^{\lambda_n t}\right)\mathbf{T}\delta\mathbf{x}_0$$

We conclude that the zero-input trajectories are linear combinations of the terms

$$\mathrm{e}^{\lambda_i t}$$

that we call <u>characteristic modes</u> or natural modes of the LTI system.

Let's analyze these terms, assuming that $\lambda_i \in \mathbb{C}$ $(\lambda_i = \alpha_i + \mathrm{j}\beta_i)$

$$\mathrm{e}^{\lambda_i t} = \mathrm{e}^{\alpha_i t}\mathrm{e}^{\mathrm{j}\beta_i t} = \mathrm{e}^{\alpha_i t}\left(\cos\left(\beta_i t\right) + \mathrm{j}\sin\left(\beta_i t\right)\right)$$

making the linear combination, the imaginary part is cancelled out by the imaginary part of the complex conjugate of λ_i.

As a consequence we have

- $\mathrm{e}^{\lambda_i t}$ when $\lambda_i \in \mathbb{R}$
- $\mathrm{e}^{\alpha_i t}\cos(\beta_i t + \varphi_i)$ when $\lambda_i \in \mathbb{C}$

Stability of LTI systems (VI)

The analysis of the two modes $\mathrm{e}^{\lambda_i t}$ and $\mathrm{e}^{\alpha_i t}\cos(\beta_i t)$ reveals that:

- if all the eigenvalues of matrix $\mathbf{A}$ lie in the open left half plane ($\alpha_i < 0$), all the modes are bounded and tend to zero asymptotically; modes tend monotonically to zero if $\beta_i = 0$, otherwise they exhibit damped oscillations
- if all the eigenvalues of matrix $\mathbf{A}$ lie in the closed left half plane ($\alpha_i \leq 0$), and there is at least one eigenvalue on the imaginary axis ($\alpha_i = 0$), all the modes are bounded but the modes associated to the eigenvalues on the imaginary axis do not tend to zero asymptotically; they are asymptotically constant if $\beta_i = 0$, otherwise they exhibit undamped oscillations
- if at least one eigenvalue of matrix $\mathbf{A}$ lies in the open right half plane ($\alpha_i > 0$), there is at least one mode that is not bounded

Stability of LTI systems (VII)

Based on the previous analysis we can conclude that an LTI system with diagonalizable state matrix is:

- asymptotically stable, if and only if all the eigenvalues of matrix $\mathbf{A}$ lie in the open left half plane ($\mathrm{Re}(\lambda_i) < 0\ \forall i$)
- stable, if and only if all the eigenvalues of matrix $\mathbf{A}$ lie in the closed left half plane ($\mathrm{Re}(\lambda_i) \leq 0\ \forall i$) and there is at least one eigenvalue on the imaginary axis ($\exists i\colon \mathrm{Re}(\lambda_i) = 0$)
- unstable, if and only if there is at least one eigenvalue of matrix $\mathbf{A}$ lying in the open right half plane ($\exists i\colon \mathrm{Re}(\lambda_i) > 0$)

This analysis can be extended to non diagonalizable matrices adopting the Jordan canonical form.

In the general case of non diagonalizable state matrices it can be shown that, if all the eigenvalues of matrix $\mathbf{A}$ lie in the closed left half plane, and there are multiple eigenvalues on the imaginary axis, the system is unstable if there is at least one eigenvalue on the imaginary axis whose geometric multiplicity is less than the algebraic multiplicity.

Stability of LTI systems (VIII)

We conclude showing that stability is a structural property of a LTI system.

Given a change of variables $\mathbf{T}$

$$\mathbf{A} \sim \hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$$

as similar matrices share eigenvalues and geometric multiplicities of eigenvalues, stability is invariant with respect to a change of state variables, and it is thus a structural property of the system.

Geometric vs. algebraic multiplicity of an eigenvalue, how are they defined? How to compute them?

Algebraic multiplicity, is the multiplicity of the eigenvalue in the characteristic equation.

Geometric multiplicity, is the number of linearly independent eigenvectors associated to the eigenvalue.

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Eigenvalues $\lambda_{1,2} = 0$

Let's compute the eigenvectors associated to the eigenvalue $\lambda = 0$

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \quad \Rightarrow \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{0} \quad \Rightarrow \quad v_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

We conclude that the eigenvalue $\lambda = 0$ has

- algebraic multiplicity 2
- geometric multiplicity 1

Stability analysis of LTI systems (I)

The stability of a LTI system is related to the eigenvalues of its state matrix, but computing the eigenvalues is, in general, not so simple (think to big matrices!).

We would like to investigate the existence of tools to perform the stability analysis without computing the eigenvalues.

Let's start observing that:

- if matrix $\mathbf{A}$ is triangular, the eigenvalues are the diagonal entries
- $\mathrm{tr}(\mathbf{A}) \geq 0$ implies that the system cannot be asymptotically stable (remember that $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mathrm{Re}(\lambda_i)$)
- $\mathrm{tr}(\mathbf{A}) > 0$ implies that the system is unstable
- $\det(\mathbf{A}) = 0$ implies that the system cannot be asymptotically stable (remember that $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$)

Consider now the characteristic polynomial of matrix $\mathbf{A}$

$$\varphi(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = \varphi_0 \lambda^n + \varphi_1 \lambda^{n-1} + \varphi_2 \lambda^{n-2} + \cdots + \varphi_n$$

We introduce a necessary, and a necessary and sufficient condition to conclude on the asymptotic stability of a LTI system analyzing the coefficients of the characteristic polynomial.

Necessary condition. If the system is asymptotically stable, then all the coefficients of the characteristic polynomial are nonzero and have the same sign.

Example

$\varphi(\lambda) = \lambda^3 + \lambda^2 - \lambda + 1$	$\varphi(\lambda) = \lambda^3 + \lambda^2 + 1$	$\varphi(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1$
not asymptotically stable	not asymptotically stable	nothing can be concluded

Stability analysis of LTI systems (III)

Consider again the characteristic polynomial of matrix $\mathbf{A}$

$$\varphi(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \varphi_0\lambda^n + \varphi_1\lambda^{n-1} + \varphi_2\lambda^{n-2} + \cdots + \varphi_n$$

We construct a table (<u>Routh table</u>) with the following rules:

- first row starts with φ_0
- second row starts with φ_1
- other coefficients alternate between rows
- both rows should be same length
  - continue until no coefficients are left
  - add zero as last coefficient if necessary
- entries in a generic row are computed following the rule

$$l_i = -\frac{1}{k_1}\det\begin{bmatrix} h_1 & h_{i+1} \\ k_1 & k_{i+1} \end{bmatrix}$$

$$\left.\begin{array}{cccc} \varphi_0 & \varphi_2 & \varphi_4 & \cdots \\ \varphi_1 & \varphi_3 & \varphi_5 & \cdots \\ \vdots & \vdots & \vdots & \\ h_1 & h_2 & h_3 & \cdots \\ k_1 & k_2 & k_3 & \cdots \\ l_1 & l_2 & l_3 & \cdots \\ \vdots & \vdots & \vdots & \end{array}\right\} n+1 \text{ rows}$$

Stability analysis of LTI systems (IV)

<u>Necessary and sufficient condition</u>. The system is asymptotically stable if and only if all entries of the first column are nonzero and have the same sign.

Example

$$\varphi(\lambda) = \lambda^3 + 4\lambda^2 + 5\lambda + 2$$

$$\begin{matrix} 1 & 5 & 0 \\ 4 & 2 & 0 \\ k_1 & k_2 & \\ h_1 & & \end{matrix}$$

$$k_1 = -\frac{1}{4}\det\begin{bmatrix}1 & 5\\ 4 & 2\end{bmatrix} = 4.5$$

$$k_2 = -\frac{1}{4}\det\begin{bmatrix}1 & 0\\ 4 & 0\end{bmatrix} = 0$$

$$h_1 = -\frac{1}{k_1}\det\begin{bmatrix}4 & 2\\ k_1 & k_2\end{bmatrix} = 2$$

$$\begin{matrix} 1 & 5 & 0 \\ 4 & 2 & 0 \\ 4.5 & 0 & \\ 2 & & \end{matrix}$$

All entries of the first column are nonzero and have the same sign.

<u>The system is asymptotically stable</u>

Why is the Routh-Hurwitz stability criterion so important if I can compute the eigenvalues with Matlab?

Example

$$\varphi(\lambda) = \lambda^4 + 2\lambda^3 + \lambda^2 + \lambda + a$$

$$\begin{matrix} 1 & 1 & a \\ 2 & 1 & 0 \\ k_1 & k_2 & \\ h_1 & h_2 & \\ l_1 & & \end{matrix}$$

$$k_1 = -\frac{1}{2}\det\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = 0.5$$

$$k_2 = -\frac{1}{2}\det\begin{bmatrix} 1 & a \\ 2 & 0 \end{bmatrix} = a$$

Why is the Routh-Hurwitz stability criterion so important if I can compute the eigenvalues with Matlab?

Example

$$\varphi(\lambda) = \lambda^4 + 2\lambda^3 + \lambda^2 + \lambda + a$$

$$\begin{array}{ccc} 1 & 1 & a \\ 2 & 1 & 0 \\ k_1 & k_2 & \\ h_1 & h_2 & \\ l_1 & & \end{array}$$

$$h_1 = -\frac{1}{k_1}\det\begin{bmatrix} 2 & 1 \\ k_1 & k_2 \end{bmatrix} = 1 - 4a$$

$$h_2 = -\frac{1}{k_1}\det\begin{bmatrix} 2 & 0 \\ k_1 & 0 \end{bmatrix} = 0$$

$$l_1 = -\frac{1}{h_1}\det\begin{bmatrix} k_1 & k_2 \\ h_1 & h_2 \end{bmatrix} = a$$

$$\begin{array}{ccc} 1 & 1 & a \\ 2 & 1 & 0 \\ 0.5 & a & \\ 1-4a & 0 & \\ a & & \end{array}$$

The system is asymptotically stable for

$$0 < a < \frac{1}{4}$$

Stability of equilibria of nonlinear systems (I)

Given a nonlinear time invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$$

and a constant input function for all $t \geq t_0$

$$\mathbf{u}(t) = \bar{\mathbf{u}} \quad t \geq t_0$$

assume that there exists a state equilibrium $\bar{\mathbf{x}}$.

How can we asses the stability of this equilibrium point?

We will exploit the stability tools for LTI systems, approximating the nonlinear system with a local linear approximation.

Let's first introduce the notion of linearized system…

Given a nonlinear time invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \qquad \mathbf{x} \in \mathbb{R}^n \quad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \qquad \mathbf{y} \in \mathbb{R}^p$$

and an equilibrium

$$\mathbf{u}(t) = \bar{\mathbf{u}} \quad \mathbf{x}(t) = \bar{\mathbf{x}} \quad \mathbf{y}(t) = \bar{\mathbf{y}} \quad \forall t$$

We can locally approximate the nonlinear system, around the equilibrium, with the linearized system

$$\delta\dot{\mathbf{x}}(t) = \mathbf{A}\delta\mathbf{x}(t) + \mathbf{B}\delta\mathbf{u}(t)$$

$$\delta\mathbf{y}(t) = \mathbf{C}\delta\mathbf{x}(t) + \mathbf{D}\delta\mathbf{u}(t)$$

where

$$\delta\mathbf{x}(t) = \mathbf{x}(t) - \bar{\mathbf{x}} \quad \delta\mathbf{u}(t) = \mathbf{u}(t) - \bar{\mathbf{u}} \quad \delta\mathbf{y}(t) = \mathbf{y}(t) - \bar{\mathbf{y}}$$

and

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \quad \mathbf{B} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \quad \mathbf{C} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \quad \mathbf{D} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$$

As the linearized system is a LTI system, we can assess the stability of the equilibrium point of the nonlinear system analyzing the state matrix

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$$

We can state the following results:

- if all the eigenvalues of matrix $\mathbf{A}$ lie in the open left half plane ($\mathrm{Re}(\lambda_i)<0$), the equilibrium point is asymptotically stable
- if at least one eigenvalue of matrix $\mathbf{A}$ lies in the open right half plane ($\exists i: \mathrm{Re}(\lambda_i)>0$), the equilibrium point is unstable

If the eigenvalues of matrix $\mathbf{A}$ lie in the closed left half plane and there is at least one eigenvalue on the imaginary axis the linearization, that is a first order approximation, is too rough to assess the stability of the equilibrium point.

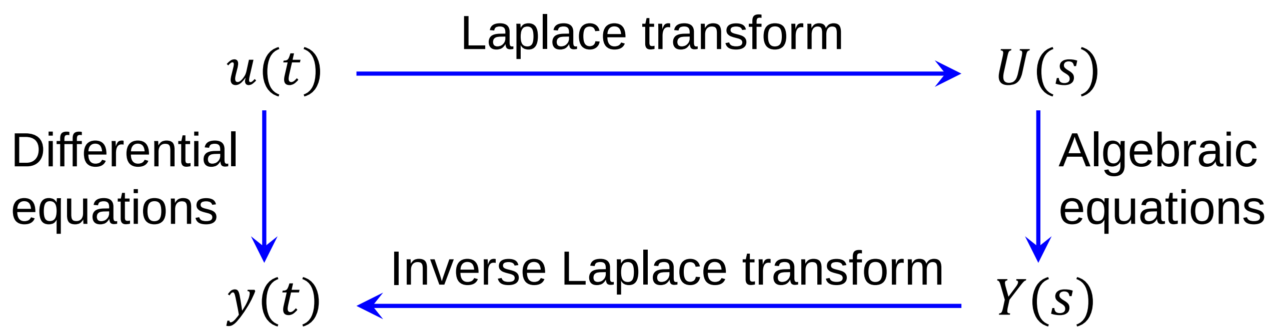
Transfer function of a LTI system (I)

Given a LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

using the Laplace transform we can introduce a representation of the system in the frequency domain.



Assuming zero initial conditions, the relation in the frequency domain is called transfer function and it is given by

$$\mathbf{G}(s) = \mathbf{C}\left(s\mathbf{I}_n - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D} \qquad \mathbf{G}(s) \in \mathbb{R}^{p \times m}$$

Transfer function of a LTI system (II)

We can easily show that the transfer function is invariant with respect to change of variables, i.e. it is a structural property of the LTI system.

Let's introduce the following change of variables

$$\hat{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t) \quad \det(\mathbf{T}) \neq 0$$

The system in the new set of state variables has the following description

$$\hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \quad \hat{\mathbf{B}} = \mathbf{T}\mathbf{B} \quad \hat{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1} \quad \hat{\mathbf{D}} = \mathbf{D}$$

Computing now the transfer function

$$\begin{aligned}
\hat{\mathbf{G}}(s) &= \hat{\mathbf{C}}\left(s\mathbf{I}_n - \hat{\mathbf{A}}\right)^{-1}\hat{\mathbf{B}} + \hat{\mathbf{D}} \\
&= \mathbf{C}\mathbf{T}^{-1}\left(s\mathbf{T}\mathbf{T}^{-1} - \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)^{-1}\mathbf{T}\mathbf{B} + \mathbf{D} \\
&= \mathbf{C}\mathbf{T}^{-1}\left[\mathbf{T}\left(s\mathbf{I}_n - \mathbf{A}\right)\mathbf{T}^{-1}\right]^{-1}\mathbf{T}\mathbf{B} + \mathbf{D} \\
&= \mathbf{C}\mathbf{T}^{-1}\mathbf{T}\left(s\mathbf{I}_n - \mathbf{A}\right)^{-1}\mathbf{T}^{-1}\mathbf{T}\mathbf{B} + \mathbf{D} \\
&= \mathbf{C}\left(s\mathbf{I}_n - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D} = \mathbf{G}(s)
\end{aligned}$$

How can we compute the transfer function of a LTI system from the state space representation?

Let's start from the LTI system in state space form

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_1u(t) \\ \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2u(t) \\ \vdots \\ \dot{x}_n(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_nu(t) \end{cases}$$

$$y(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t) + du(t)$$

transforming each equation we obtain

$$\begin{cases} sX_1(s) = a_{11}X_1(s) + a_{12}X_2(s) + \cdots + a_{1n}X_n(s) + b_1U(s) \\ sX_2(s) = a_{21}X_1(s) + a_{22}X_2(s) + \cdots + a_{2n}X_n(s) + b_2U(s) \\ \vdots \\ sX_n(s) = a_{n1}X_1(s) + a_{n2}X_2(s) + \cdots + a_{nn}X_n(s) + b_nU(s) \end{cases}$$

$$Y(s) = c_1X_1(s) + c_2X_2(s) + \cdots + c_nX_n(s) + dU(s)$$

Transfer function of a LTI system (IV)

$$\begin{cases} sX_1(s) = a_{11}X_1(s) + a_{12}X_2(s) + \cdots + a_{1n}X_n(s) + b_1U(s) \\ sX_2(s) = a_{21}X_1(s) + a_{22}X_2(s) + \cdots + a_{2n}X_n(s) + b_2U(s) \\ \vdots \\ sX_n(s) = a_{n1}X_1(s) + a_{n2}X_2(s) + \cdots + a_{nn}X_n(s) + b_nU(s) \end{cases}$$

$$Y(s) = c_1X_1(s) + c_2X_2(s) + \cdots + c_nX_n(s) + dU(s)$$

Solving now this linear system we get the transfer function

$$G(s) = \frac{Y(s)}{U(s)}$$

We will now analyze the structure of the transfer function.

The transfer function is a ratio of polynomials in the s variable

$$G(s) = \frac{N(s)}{D(s)}$$

Numerator
- proper system, polynomial of order n
- strictly proper system, polynomial of order $< n$

Denominator
- polynomial of order n
- is the characteristic polynomial of matrix $\mathbf{A}$

We will call
- poles, the roots of the denominator
- zeros, the roots of the numerator

The stability of the system can be assessed analyzing the poles!

Caveat: these conclusions hold only if there are no pole-zero cancellations.

Transfer function of a LTI system (VI)

Let's introduce two standard ways to represent a transfer function.

First, the zero-pole form

$$G(s) = \rho \frac{\prod_i (s+z_i)}{\prod_j (s+p_j)}$$

where

- $\rho \in \mathbb{R}$ is the gain
- $-z_i \in \mathbb{C}$ are the zeros
- $-p_j \in \mathbb{C}$ are the poles

In order to have polynomials with real coefficients we can modify the representation as follows

$$G(s) = \rho \frac{\prod_i (s+z_i) \prod_i \left(s^2 + 2\xi_{z_i} \omega_{n_{z_i}} s + \omega_{n_{z_i}}^2\right)}{\prod_j (s+p_j) \prod_j \left(s^2 + 2\xi_{p_j} \omega_{n_{p_j}} s + \omega_{n_{p_j}}^2\right)}$$

Transfer function of a LTI system (VII)

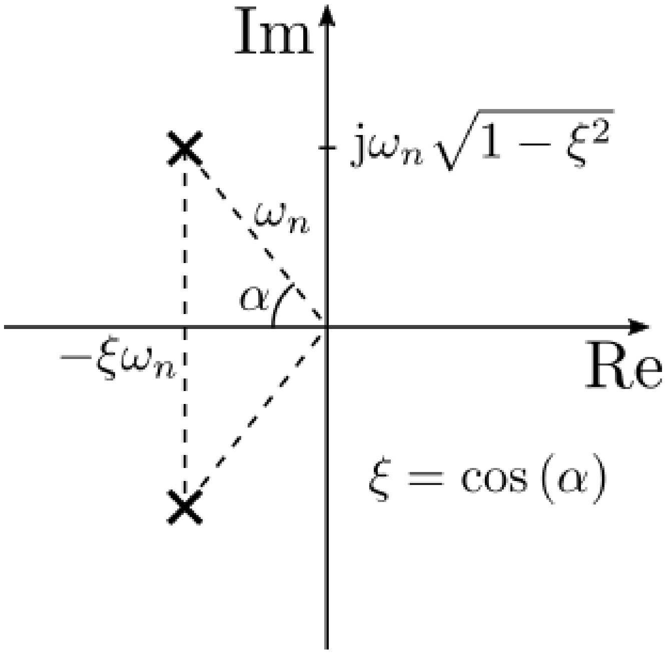
$$G(s) = \rho \frac{\prod_i (s+z_i) \prod_i \left(s^2 + 2\xi_{z_i}\omega_{n_{z_i}} s + \omega_{n_{z_i}}^2\right)}{\prod_j (s+p_j) \prod_j \left(s^2 + 2\xi_{p_j}\omega_{n_{p_j}} s + \omega_{n_{p_j}}^2\right)}$$

where

- $\omega_{n_{z_i}}, \omega_{n_{p_j}}$ are positive real numbers called natural frequencies
- ξ_{z_i}, ξ_{p_j} are real numbers, with $|\xi_*| < 1$, called damping factors

They can be interpreted as follows

- $\mathrm{Re}(-p_i) = -\xi_{p_i}\omega_{n_{p_i}}$
- $\mathrm{Im}(-p_i) = \omega_{n_{p_i}}\sqrt{1-\xi_{p_i}^2}$



A second form is the gain-time constant form

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + s\tau_i)}{\prod_j (1 + sT_j)}$$

where

- $\mu \in \mathbb{R}$ is the gain
- $g \in \mathbb{Z}$ is the type (number of poles/zeros in $s = 0$)
- $\tau_i \in \mathbb{C}$ are the zero time constants
- $T_j \in \mathbb{C}$ are the pole time constants

In order to have polynomials with real coefficients we can modify the representation as follows

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + s\tau_i) \prod_i \left(1 + 2\frac{\xi_{z_i}}{\omega_{n_{z_i}}} s + \frac{s^2}{\omega_{n_{z_i}}^2}\right)}{\prod_j (1 + sT_j) \prod_j \left(1 + 2\frac{\xi_{p_j}}{\omega_{n_{p_j}}} s + \frac{s^2}{\omega_{n_{p_j}}^2}\right)}$$

Transfer function of a LTI system (IX)

$$G(s) = \frac{\mu}{s^g} \frac{\prod_i (1 + s\tau_i) \prod_i \left(1 + 2\frac{\xi_{z_i}}{\omega_{nz_i}} s + \frac{s^2}{\omega_{nz_i}^2}\right)}{\prod_j (1 + sT_j) \prod_j \left(1 + 2\frac{\xi_{p_j}}{\omega_{np_j}} s + \frac{s^2}{\omega_{np_j}^2}\right)}$$

If $g = 0$ (no poles/zeros in $s = 0$) then

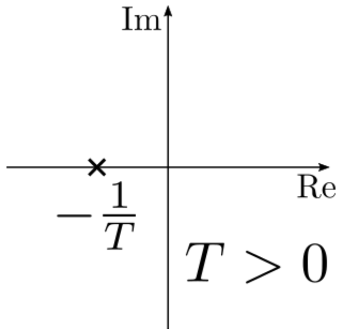
$$\mu = \lim_{s \to 0} G(s) = G(0) = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + D = \mu_s$$

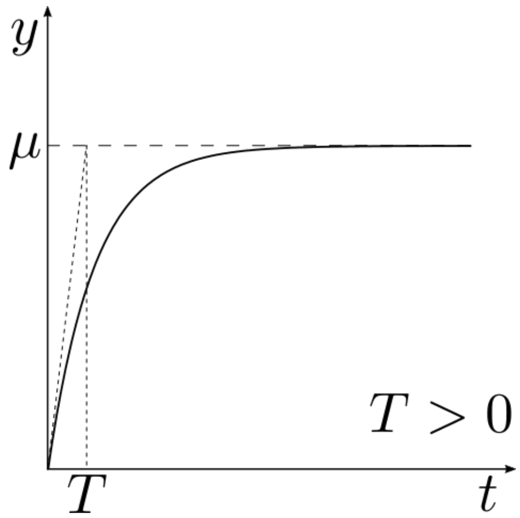
and the gain of the transfer function is equal to the system static gain.

Transfer function of a LTI system – Examples (I)

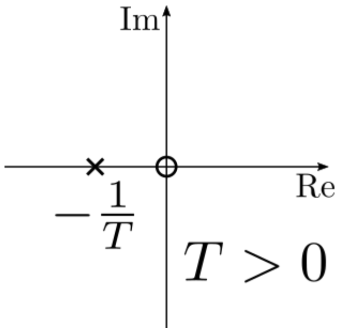
Step response of first order systems

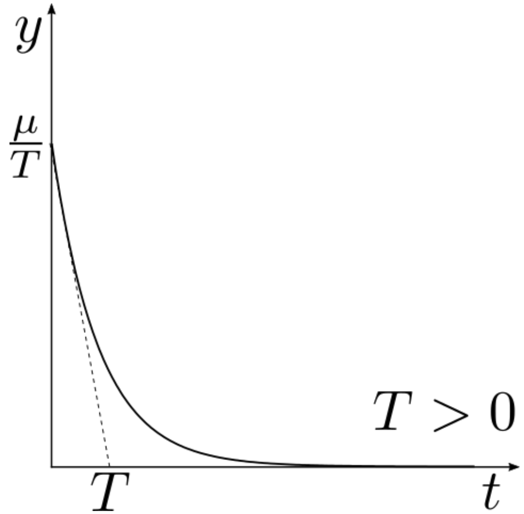
$$G(s) = \frac{\mu}{1+sT}$$





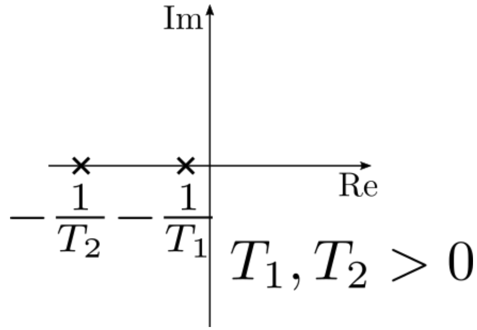
$$G(s) = \mu \frac{s}{1+sT}$$

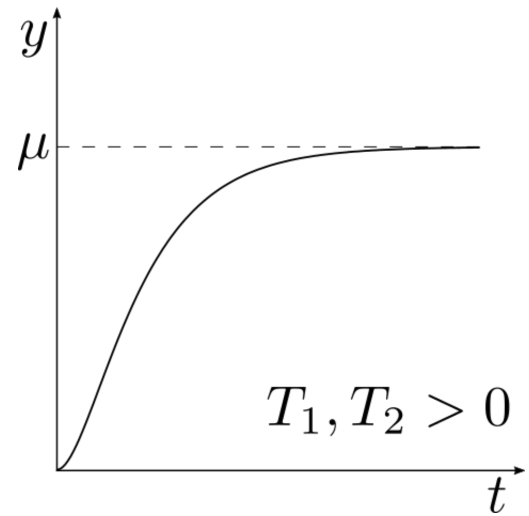




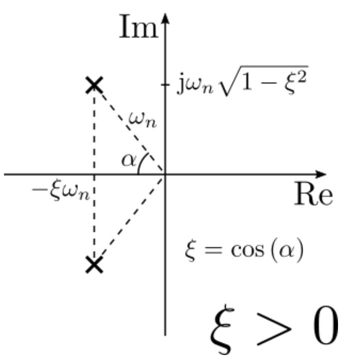
Step response of second order systems

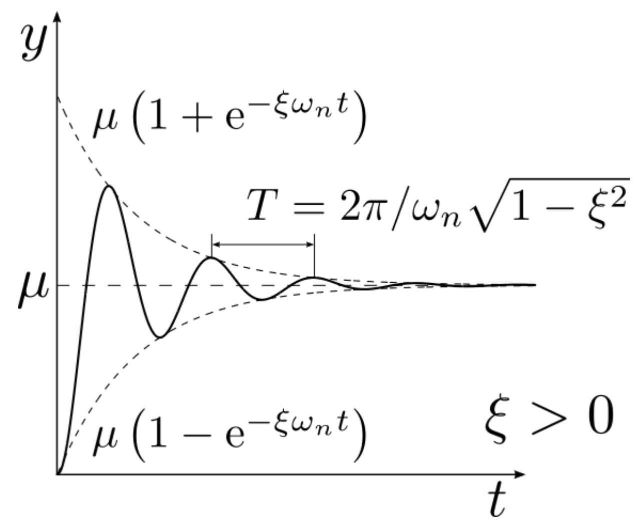
$$G(s) = \frac{\mu}{(1+sT_1)(1+sT_2)}$$





$$G(s) = \mu \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$





Up to now we haven't considered pole-zero cancellations that can occur during the computation of the transfer function from the state space matrices.

Let's see two examples.

$$\begin{cases} \dot{x}_1 = -2x_1 \\ \dot{x}_2 = x_1 - x_2 + u \\ y = 2x_1 + 3x_2 \end{cases} \Rightarrow \quad G(s) = \cdots = \frac{3(s+2)}{(s+2)(s+1)} = \frac{3}{s+1}$$

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + u \\ \dot{x}_2 = -x_2 + u \\ y = x_2 \end{cases} \Rightarrow \quad G(s) = \cdots = \frac{s+1}{(s+1)^2} = \frac{1}{s+1}$$

In both examples the order of the denominator is less than the order of the state space representation: there are eigenvalues of matrix $\mathbf{A}$ that are not poles of the transfer function.

The poles that have been canceled out form the hidden dynamics.

Observability and controllability (II)

We can formalize the fact that there are parts of the system that do not appear in the input-output relation introducing the notions of observability and controllability.

A LTI system is completely controllable if we can find an input function that, in finite time, can move the state of the system from the origin of the state space to any point in $\mathbb{R}^n$.

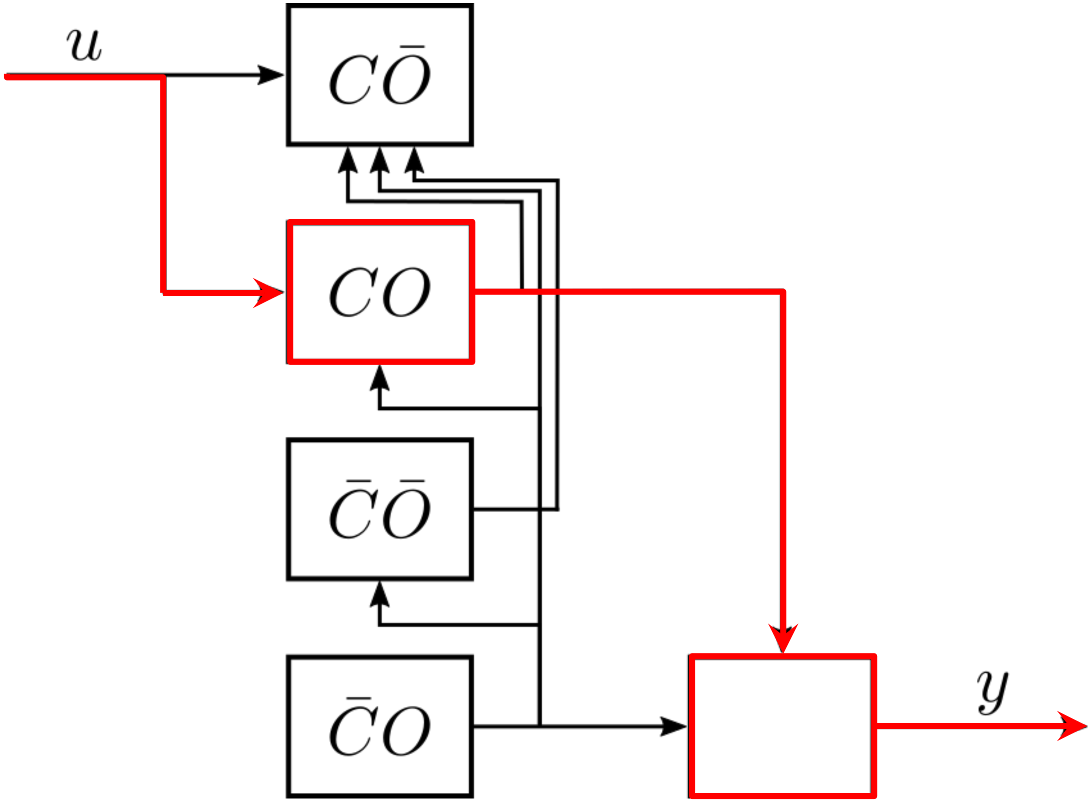
A LTI system is completely observable if for any initial condition in $\mathbb{R}^n$ the zero-input response of the system output, on any finite time interval, is different from a zero output.

If a LTI system is not completely controllable and not completely observable, it means that the system has a part that is observable but not controllable, another that is controllable but not observable, etc.

We can always find a change of variables that puts the system in a canonical form (Kalman canonical form) that separates and shows the different parts.

Observability and controllability (III)

Here is the results of the Kalman canonical decomposition



The only block that appears in the path from input u to output y is the completely controllable and observable part.

The transfer function is thus the image of only the completely controllable and observable part of the system.

Observability and controllability (IV)

How can we assess the observability/controllability of a LTI system?

We introduce the controllability matrix $\mathbf{K}_c$ ($\mathbf{K}_c \in \mathbb{R}^{n\times nm}$)

$$\mathbf{K}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

and the observability matrix $\mathbf{K}_o$ ($\mathbf{K}_o \in \mathbb{R}^{n\times np}$)

$$\mathbf{K}_o = \begin{bmatrix} \mathbf{C}^T & \mathbf{A}^T\mathbf{C}^T & \mathbf{A}^{T^2}\mathbf{C}^T & \cdots & \mathbf{A}^{T^{n-1}}\mathbf{C}^T \end{bmatrix}$$

A LTI system is completely controllable if and only if $\mathrm{rank}(\mathbf{K}_c) = n$.
A LTI system is completely observable if and only if $\mathrm{rank}(\mathbf{K}_o) = n$.

If the system is single input (single output) the controllability (observability) condition is equivalent to verify that $\mathbf{K}_c$ ($\mathbf{K}_o$) is a non-singular matrix.

Observability and controllability – Examples (I)

Example 1

$$\begin{cases} \dot{x}_1 & = & -2x_1 \\ \dot{x}_2 & = & x_1 - x_2 + u \\ y & = & 2x_1 + 3x_2 \end{cases} \quad \Rightarrow \quad G(s) = \cdots = \frac{3(s+2)}{(s+2)(s+1)} = \frac{3}{s+1}$$

$$\mathbf{K}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{K}_c) = 0$$

not completely controllable

$$\mathbf{K}_o = \begin{bmatrix} \mathbf{C}^T & \mathbf{A}^T\mathbf{C}^T \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -3 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{K}_o) \neq 0$$

completely observable

The system is composed of two parts: one is completely observable and controllable, another one is completely observable but it is not completely controllable.

Example 2

$$\begin{cases} \dot{x}_1 &= -x_1 + x_2 + u \\ \dot{x}_2 &= -x_2 + u \\ y &= x_2 \end{cases} \quad \Rightarrow \quad G(s) = \cdots = \frac{s+1}{(s+1)^2} = \frac{1}{s+1}$$

$$\mathbf{K}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{K}_c) \neq 0$$

completely controllable

$$\mathbf{K}_o = \begin{bmatrix} \mathbf{C}^T & \mathbf{A}^T\mathbf{C}^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{K}_o) = 0$$

not completely observable

The system is composed of two parts: one is completely observable and controllable, another one is completely controllable but it is not completely observable.

We can easily show that observability and controllability are invariant with respect to change of variables, i.e. they are structural properties of the LTI system.

Let's introduce the following change of variables

$$\hat{\mathbf{x}}(t) = \mathbf{T}\mathbf{x}(t) \qquad \det(\mathbf{T}) \neq 0$$

The system in the new set of state variables has the following description

$$\hat{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} \quad \hat{\mathbf{B}} = \mathbf{T}\mathbf{B} \quad \hat{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1} \quad \hat{\mathbf{D}} = \mathbf{D}$$

Computing now the observability matrix

$$\begin{aligned}
\hat{\mathbf{K}}_o &= \begin{bmatrix} \hat{\mathbf{C}}^T & \hat{\mathbf{A}}^T\hat{\mathbf{C}}^T & \hat{\mathbf{A}}^{T^2}\hat{\mathbf{C}}^T & \cdots & \hat{\mathbf{A}}^{T^{n-1}}\hat{\mathbf{C}}^T \end{bmatrix} \\
&= \big[\mathbf{T}^{-T}\mathbf{C}^T \quad \mathbf{T}^{-T}\mathbf{A}^T\mathbf{T}^T\mathbf{T}^{-T}\mathbf{C}^T \quad \mathbf{T}^{-T}\mathbf{A}^T\mathbf{T}^T\mathbf{T}^{-T}\mathbf{A}^T\mathbf{T}^T\mathbf{T}^{-T}\mathbf{C}^T \\
&\qquad \cdots \quad \left(\mathbf{T}^{-T}\mathbf{A}^T\mathbf{T}^T\mathbf{T}^{-T}\mathbf{A}^T\mathbf{T}^T \ldots \mathbf{T}^{-T}\mathbf{A}^T\mathbf{T}^T\right)\mathbf{T}^{-T}\mathbf{C}^T\big] \\
&= \begin{bmatrix} \mathbf{T}^{-T}\mathbf{C}^T & \mathbf{T}^{-T}\mathbf{A}^T\mathbf{C}^T & \mathbf{T}^{-T}\mathbf{A}^{T^2}\mathbf{C}^T & \cdots & \mathbf{T}^{-T}\mathbf{A}^{T^{n-1}}\mathbf{C}^T \end{bmatrix} \\
&= \mathbf{T}^{-T}\begin{bmatrix} \mathbf{C}^T & \mathbf{A}^T\mathbf{C}^T & \mathbf{A}^{T^2}\mathbf{C}^T & \cdots & \mathbf{A}^{T^{n-1}}\mathbf{C}^T \end{bmatrix} = \mathbf{T}^{-T}\mathbf{K}_o
\end{aligned}$$

and the controllability matrix

$$\begin{aligned}\hat{\mathbf{K}}_c &= \left[\hat{\mathbf{B}} \quad \hat{\mathbf{A}}\hat{\mathbf{B}} \quad \hat{\mathbf{A}}^2\hat{\mathbf{B}} \quad \cdots \quad \hat{\mathbf{A}}^{n-1}\hat{\mathbf{B}}\right] \\ &= \left[\mathbf{TB} \quad \mathbf{TAT}^{-1}\mathbf{TB} \quad \mathbf{TAT}^{-1}\mathbf{TAT}^{-1}\mathbf{TB} \right. \\ &\qquad \left. \cdots \quad \left(\mathbf{TAT}^{-1}\mathbf{TAT}^{-1}\ldots\mathbf{TAT}^{-1}\right)\mathbf{TB}\right] \\ &= \left[\mathbf{TB} \quad \mathbf{TAB} \quad \mathbf{TA}^2\mathbf{B} \quad \cdots \quad \mathbf{TA}^{n-1}\mathbf{B}\right] \\ &= \mathbf{T}\left[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}\right] = \mathbf{TK}_c\end{aligned}$$

Summarizing

$$\hat{\mathbf{K}}_o = \mathbf{T}^{-T}\mathbf{K}_o \qquad \hat{\mathbf{K}}_c = \mathbf{TK}_c$$

As matrix $\mathbf{T}$ is non-singular, we conclude that $\widehat{\mathbf{K}}_o$ and $\mathbf{K}_o$ ($\widehat{\mathbf{K}}_c$ and $\mathbf{K}_c$) share the same rank.

Consequently the system in the new state variables is completely observable (controllable) if and only if the original system is completely observable (controllable).

Realization and canonical forms (I)

Realization refers to the computation of a state space model implementing a given input-output behavior.

We observe that:

- the realization of an input-output relation has not a unique solution
- the solutions characterized by an order of the state space model equal to the order of the denominator of the transfer function are called minimal realizations
- minimal realizations of an input-output behavior are LTI systems completely controllable and completely observable

We now introduce two canonical realizations.

Realization and canonical forms (II)

Given the transfer function of a SISO strictly proper LTI system

$$G(s) = \frac{b_n s^{n-1} + b_{n-1} s^{n-2} + \cdots + b_2 s + b_1}{s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_2 s + a_1}$$

The controllable canonical form is given by the following matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}$$

The resulting model is guaranteed to be completely controllable.

Realization and canonical forms (III)

Given the transfer function of a SISO strictly proper LTI system

$$G(s) = \frac{b_n s^{n-1} + b_{n-1} s^{n-2} + \cdots + b_2 s + b_1}{s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_2 s + a_1}$$

The observable canonical form is given by the following matrices

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_n \end{bmatrix} \qquad \tilde{\mathbf{B}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$\tilde{\mathbf{C}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

The resulting model is guaranteed to be completely observable.

Realization and canonical forms (IV)

Given a SISO strictly proper LTI system, described by matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, we have

$$G(s) = \mathbf{C}\left(s\mathbf{I}_n - \mathbf{A}\right)^{-1}\mathbf{B} = G(s)^T = \left[\mathbf{C}\left(s\mathbf{I}_n - \mathbf{A}\right)^{-1}\mathbf{B}\right]^T$$

$$= \mathbf{B}^T\left(s\mathbf{I}_n - \mathbf{A}\right)^{-T}\mathbf{C}^T = \mathbf{B}^T\left(s\mathbf{I}_n - \mathbf{A}^T\right)^{-1}\mathbf{C}^T$$

The system described by matrices

$$\tilde{\mathbf{A}} = \mathbf{A}^T \quad \tilde{\mathbf{B}} = \mathbf{C}^T \quad \tilde{\mathbf{C}} = \mathbf{B}^T$$

has the same transfer function

$$G(s) = \tilde{\mathbf{C}}\left(s\mathbf{I}_n - \tilde{\mathbf{A}}\right)^{-1}\tilde{\mathbf{B}}$$

of the original system, and is called dual system.

The original system and the dual one are two realizations of the same input-output relation.

Caveat: The observable canonical form is the dual system of the controllable canonical form!

Given a completely controllable LTI system, how can we compute the change of variables that transforms its state space description into the controllable canonical form?

Given $(\mathbf{A}, \mathbf{B})$, completely controllable, follow the steps:

1. compute the characteristic polynomial of matrix $\mathbf{A}$
$$\det(s\mathbf{I}-\mathbf{A}) = s^n + a_n s^{n-1} + a_{n-1} s^{n-2} + \cdots + a_2 s + a_1$$
2. using coefficients a_i, compute $(\widehat{\mathbf{A}}, \widehat{\mathbf{B}})$
3. compute the controllability matrices $\mathbf{K}_c$ and $\widehat{\mathbf{K}}_c$
4. compute
$$\mathbf{T} = \hat{\mathbf{K}}_c \mathbf{K}_c^{-1}$$

Given

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

compute the change of variables that puts the system in controllable canonical form.

1. compute the characteristic polynomial of matrix $\mathbf{A}$

$$\det(s\mathbf{I} - \mathbf{A}) = \begin{vmatrix} s+1 & -1 & 0 \\ 1 & s & -1 \\ -1 & 0 & s+2 \end{vmatrix} = (s+1)\left(s^2+2s\right) + (s+2-1) = s^3+3s^2+3s+1$$

2. compute $(\widehat{\mathbf{A}}, \widehat{\mathbf{B}})$

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \quad \hat{\mathbf{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Realization and canonical forms – Example (II)

3. compute the controllability matrices $\mathbf{K}_c$ and $\widehat{\mathbf{K}}_c$

$$\mathbf{K}_c = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix} \quad \hat{\mathbf{K}}_c = \begin{bmatrix} \hat{\mathbf{B}} & \hat{\mathbf{A}}\hat{\mathbf{B}} & \hat{\mathbf{A}}^2\hat{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 6 \end{bmatrix}$$

4. compute the change of variables

$$\mathbf{T} = \hat{\mathbf{K}}_c \mathbf{K}_c^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 6 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$