Control of Mobile Robots

Dynamics of mobile robots

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Now that we have introduced all the tools required to develop the kinematic model of a general mobile robot, we would like to study how the motion of the robot is related to the generalized forces acting on it, i.e., the dynamic model.

The main topics on dynamic modelling are

- dynamics review
- dynamic model of a mobile robot
    - Lagrange formulation
    - Newton-Euler formulation
- dynamic model of a unicycle and a bicycle robot
- tire, wheel and actuator modelling

When should we adopt a dynamic instead of a kinematic model?

- to set up an accurate simulator for control system testing and validation
- to design a control system when the effects on forces acting on the system play a relevant role
    - high speed / high acceleration manoeuvers
    - driving at the limit of handling
    - significant mass / inertia changes
    - significant load transfer
    - …

In this part we review the Lagrange and Newton-Euler modelling tools, focusing on the dynamic models of unicycle and bicycle robots.

Dynamics review – Elementary and virtual displacements

We start again reviewing fundamental tools of rigid-body dynamics.

The motion of a system with n DOF and subjected to holonomic bilateral (equality) constraints is described by

$$\mathbf{x} = \mathbf{x}\left(\mathbf{q}\left(t\right), t\right)$$

The *elementary displacement* in the interval $(t, t + \mathrm{d}t)$ is

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}\left(\mathbf{q}, t\right)}{\partial \mathbf{q}}\dot{\mathbf{q}}\mathrm{d}t + \frac{\partial \mathbf{x}\left(\mathbf{q}, t\right)}{\partial t}\mathrm{d}t$$

An elementary displacement is an actual motion of the system in the interval $(t, t + \mathrm{d}t)$ consistent with constraints

The *virtual displacement* at time t due to an increment $\delta \boldsymbol{q}$ is instead given by

$$\delta\mathbf{x} = \frac{\partial \mathbf{x}\left(\mathbf{q}, t\right)}{\partial \mathbf{q}}\delta\mathbf{q}$$

A virtual displacement is an imaginary motion of the system when constraints are made invariant and equal to those at time t

If the constraints are holonomic and scleronomic (time invariant)

$$\mathbf{x} = \mathbf{x}(\mathbf{q}(t))$$

then $\delta\boldsymbol{q} = \mathrm{d}\boldsymbol{q} = \dot{\boldsymbol{q}}\mathrm{d}t$ and virtual displacements coincide with elementary displacements.

If a system of forces is applied to the body, to a virtual displacement can be associated a *virtual work*. As a direct consequence of Newton laws of motion

$$\delta W_m + \delta W_a + \delta W_h = 0$$

Virtual work done by inertia forces

Virtual work done by active forces

Virtual work done by reaction forces

$$\cancel{\delta W_m} + \delta W_a + \cancel{\delta W_h} = 0$$

At steady state, inertia forces are null

In the case of frictionless equality constraints

The equilib ius that the virtual work of t entically null on any ent

$$\delta W_a = 0$$

This is known as the *principle of virtual work* and is the fundamental equation of statics of a constrained system.

Denoting with ζ the active generalized forces, the virtual work can be expressed as

$$\delta W_a = \zeta^T \delta \mathbf{q} = 0$$

In the dynamic case we can distinguish active forces into

- *conservative forces*
- *non-conservative forces*

The work done by the force is independent of the trajectory described by the point of application, it depends only on the initial and final position of the point of application

The virtual work of conservative forces is given by

$$\delta W_c = -\frac{\partial \mathcal{U}}{\partial \mathbf{q}} \boldsymbol{\delta} \mathbf{q}$$

where $\mathcal{U}(\boldsymbol{q})$ is the total potential energy of the system.

The virtual work of non conservative forces is expressed in the form

$$\delta W_{nc} = \xi^T \boldsymbol{\delta} \mathbf{q}$$

where ξ is the vector of non conservative forces.

The vector of generalized forces can be thus expressed as

$$\zeta = \xi - \left(\frac{\partial \mathcal{U}}{\partial \mathbf{q}} \right)^T$$

The virtual work associated to the inertial forces can be computed from the kinetic energy as follows

$$\delta W_m = \left(\frac{\partial \mathcal{T}}{\partial \mathbf{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{T}}{\partial \dot{\mathbf{q}}} \right) \delta \mathbf{q}$$

Summarizing

$$\delta W_m + \delta W_a = 0$$

$$\delta W_a = \left(\xi - \left(\frac{\partial \mathcal{U}}{\partial \mathbf{q}} \right)^T \right)^T \delta \mathbf{q}$$

we obtain the ***Lagrange equations***

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \right)^T - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \right)^T = \xi$$

where

$$\mathcal{L} = \mathcal{T} - \mathcal{U}$$

is the ***Lagrangian function*** of the system.

Dynamic model of a mobile robot

The dynamic model can be derived following a general procedure similar to the one used for manipulators, taking into account nonholonomic constraints!

We start defining the Lagrangian of the system

$$\mathscr{L}(\mathbf{q},\dot{\mathbf{q}}) = \mathscr{T}(\mathbf{q},\dot{\mathbf{q}}) - \mathscr{U}(\mathbf{q}) = \frac{1}{2}\dot{\mathbf{q}}^T B(\mathbf{q})\dot{\mathbf{q}} - \mathscr{U}(\mathbf{q})$$

Symmetric and positive definite *inertia matrix*

The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial \mathscr{L}}{\partial \mathbf{q}}\right)^T = S(\mathbf{q})\,\tau + A(\mathbf{q})\,\lambda$$

Vector of reaction forces

Vector of k *Lagrange multipliers*

Mapping from the external inputs to generalized forces performing work on $\boldsymbol{q}$

$n-k$ external inputs

Matrix representing the k-kinematic constraints

Dynamic model of a mobile robot

From the Lagrange equations we obtain

$$B(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q},\dot{\mathbf{q}}) = S(\mathbf{q})\boldsymbol{\tau} + A(\mathbf{q})\boldsymbol{\lambda}$$

$$A^T(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$$

where

$$\mathbf{n}(\mathbf{q},\dot{\mathbf{q}}) = \dot{B}(\mathbf{q})\dot{\mathbf{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\left(\dot{\mathbf{q}}^T B(\mathbf{q})\dot{\mathbf{q}}\right)\right)^T + \left(\frac{\partial \mathscr{U}(\mathbf{q})}{\partial \mathbf{q}}\right)^T$$

The kinematic constraints can be substituted with the kinematic model

$$\dot{\mathbf{q}} = G(\mathbf{q})\mathbf{v} = \sum_{i=1}^{m} \mathbf{g}_i(\mathbf{q}) v_i$$

and the Lagrange equations can be simplified multiplying by $G^T(\boldsymbol{q})$

$$G^T(\mathbf{q})\left(B(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q},\dot{\mathbf{q}})\right) = G^T(\mathbf{q})S(\mathbf{q})\boldsymbol{\tau}$$

Differentiating the kinematic model we obtain

$$\ddot{\mathbf{q}} = \dot{G}(\mathbf{q})\mathbf{v} + G(\mathbf{q})\dot{\mathbf{v}}$$

Pre-multiplying by $G^T(\boldsymbol{q})B(\boldsymbol{q})$

$$G^T(\mathbf{q})B(\mathbf{q})\ddot{\mathbf{q}} = G^T(\mathbf{q})B(\mathbf{q})\dot{G}(\mathbf{q})\mathbf{v} + G^T(\mathbf{q})B(\mathbf{q})G(\mathbf{q})\dot{\mathbf{v}}$$

and introducing the model

$$G^T(\mathbf{q})\left(B(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q},\dot{\mathbf{q}})\right) = G^T(\mathbf{q})S(\mathbf{q})\boldsymbol{\tau}$$

we obtain

$$\boxed{G^T(\mathbf{q})B(\mathbf{q})G(\mathbf{q})}\dot{\mathbf{v}} + \boxed{G^T(\mathbf{q})B(\mathbf{q})\dot{G}(\mathbf{q})\mathbf{v} + G^T(\mathbf{q})\mathbf{n}(\mathbf{q},\dot{\mathbf{q}})} = G^T(\mathbf{q})S(\mathbf{q})\boldsymbol{\tau}$$

$$M(\mathbf{q}) = G^T(\mathbf{q})B(\mathbf{q})G(\mathbf{q}) \qquad m(\mathbf{q},\mathbf{v}) = G^T(\mathbf{q})B(\mathbf{q})\dot{G}(\mathbf{q})\mathbf{v} + G^T(\mathbf{q})\mathbf{n}(\mathbf{q},G(\mathbf{q})\mathbf{v})$$

and finally

$$M(\mathbf{q})\dot{\mathbf{v}} + \mathbf{m}(\mathbf{q},\mathbf{v}) = G^T(\mathbf{q})S(\mathbf{q})\boldsymbol{\tau}$$

Dynamic model of a mobile robot

$$M(\mathbf{q})\dot{\mathbf{v}} + \mathbf{m}(\mathbf{q},\mathbf{v}) = G^T(\mathbf{q})S(\mathbf{q})\tau$$

where

$$M(\mathbf{q}) = G^T(\mathbf{q})B(\mathbf{q})G(\mathbf{q}) \qquad m(\mathbf{q},\mathbf{v}) = G^T(\mathbf{q})B(\mathbf{q})\dot{G}(\mathbf{q})\mathbf{v} + G^T(\mathbf{q})\mathbf{n}(\mathbf{q},G(\mathbf{q})\mathbf{v})$$

A few remarks:

- $M(\boldsymbol{q})$ is positive definite (remember that $B(\boldsymbol{q})$ is always positive definite!)
- $\dot{G}(\boldsymbol{q})\boldsymbol{v}$ can be derived from the time differentiation of the kinematic model

$$\dot{G}(\mathbf{q})\mathbf{v} = \left\{ v_1 \begin{bmatrix} \frac{\partial g_{11}(\mathbf{q})}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial g_{n1}(\mathbf{q})}{\partial \mathbf{q}} \end{bmatrix} + v_2 \begin{bmatrix} \frac{\partial g_{12}(\mathbf{q})}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial g_{n2}(\mathbf{q})}{\partial \mathbf{q}} \end{bmatrix} + \cdots + v_m \begin{bmatrix} \frac{\partial g_{1m}(\mathbf{q})}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial g_{nm}(\mathbf{q})}{\partial \mathbf{q}} \end{bmatrix} \right\} G(\mathbf{q})\mathbf{v}$$

See additional material A for the derivation

Dynamic model of a mobile robot

Let's summarize the dynamic model equations

$$\dot{\mathbf{q}} = G(\mathbf{q})\,\mathbf{v}$$

$$M(\mathbf{q})\,\dot{\mathbf{v}} + \mathbf{m}(\mathbf{q},\mathbf{v}) = G^T(\mathbf{q})\,S(\mathbf{q})\,\boldsymbol{\tau}$$

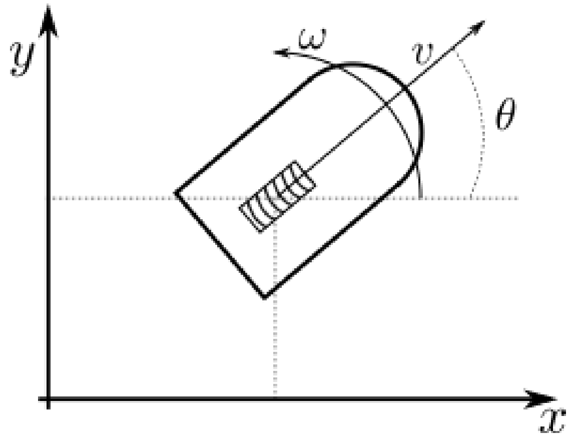
we can recast it into a state-space model ($n + m$ differential equations)

$$\dot{\mathbf{q}} = G(\mathbf{q})\,\mathbf{v}$$

$$\dot{\mathbf{v}} = -M^{-1}(\mathbf{q})\,\mathbf{m}(\mathbf{q},\mathbf{v}) + M^{-1}(\mathbf{q})\,G^T(\mathbf{q})\,S(\mathbf{q})\,\boldsymbol{\tau}$$

Let's use this formulation to derive the dynamic model of a unicycle robot...

Dynamic model of a unicycle



The unicycle robot is characterized by

- mass m
- moment of inertia I with respect to the vertical axis through its center

The configuration of the robot is described by

$$\mathbf{q} = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$$

The inertia matrix can be defined as

$$B(\mathbf{q}) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix}$$

We now have to compute $\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})$

$$\mathbf{n}(\mathbf{q},\dot{\mathbf{q}}) = \dot{B}(\mathbf{q})\,\dot{\mathbf{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \mathbf{q}}\left(\dot{\mathbf{q}}^T B(\mathbf{q})\,\dot{\mathbf{q}}\right)\right)^T + \left(\frac{\partial \mathscr{U}(\mathbf{q})}{\partial \mathbf{q}}\right)^T$$

$B(\boldsymbol{q})$ is constant

$B(\boldsymbol{q})$ does not depend on $\boldsymbol{q}$

$\mathcal{U}(\boldsymbol{q}) = 0$

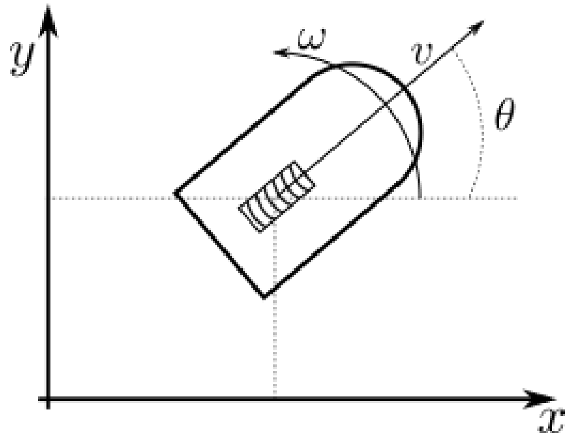
as a consequence $\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = 0$

Let’s … l from t… analysis

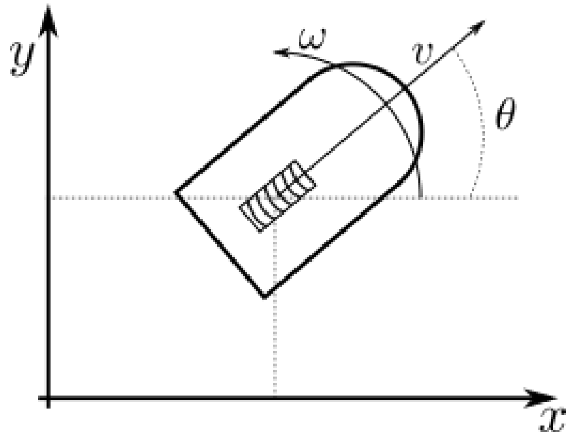
$$A^T(\mathbf{q}) = \begin{bmatrix}\sin(\theta) & -\cos(\theta) & 0\end{bmatrix}$$

and

$$G(\mathbf{q}) = \begin{bmatrix}\cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1\end{bmatrix}$$



Let's now put everything together, starting from the general formulation of the dynamic model



$$\dot{\mathbf{q}} = G(\mathbf{q})\,\mathbf{v}$$

$$M(\mathbf{q})\,\dot{\mathbf{v}} + \mathbf{m}(\mathbf{q},\mathbf{v}) = G^T(\mathbf{q})\,S(\mathbf{q})\,\tau$$

τ is the input vector, for the unicycle model is given by

- τ_1 the driving force
- τ_2 the steering torque

$S(\boldsymbol{q})$ maps the inputs to the generalized forces performing work on $\boldsymbol{q}$, and is thus given by

$$S(\mathbf{q}) = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

We thus conclude

$$G^T(\mathbf{q})S(\mathbf{q}) = G^T(\mathbf{q})G(\mathbf{q}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$G^T(\mathbf{q})B(\mathbf{q})G(\mathbf{q}) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}$$

Finally

$$\dot{G}(\mathbf{q})\mathbf{v} = \left\{ \begin{bmatrix} 0 & 0 & -\sin(\theta) \\ 0 & 0 & \cos(\theta) \\ 0 & 0 & 0 \end{bmatrix} v_1 + \mathbf{0}\, v_2 \right\} G(\mathbf{q})\mathbf{v} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} v_1 v_2$$

As a consequence

$$G^T(\mathbf{q})B(\mathbf{q})\dot{G}(\mathbf{q})\mathbf{v} = \begin{bmatrix}\cos(\theta) & \sin(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix}v_1v_2$$

$$= \begin{bmatrix}m\cos(\theta) & m\sin(\theta) & 0\\ 0 & 0 & I\end{bmatrix}\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix}v_1v_2 = \begin{bmatrix}0\\ 0\end{bmatrix}$$

We can now conclude

$$M(\mathbf{q}) = G^T(\mathbf{q})B(\mathbf{q})G(\mathbf{q}) = \begin{bmatrix}m & 0\\ 0 & I\end{bmatrix}$$

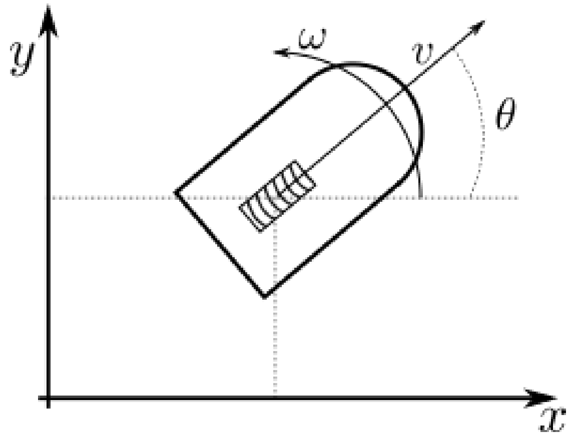
$$\mathbf{m}(\mathbf{q},\mathbf{v}) = G^T(\mathbf{q})B(\mathbf{q})\dot{G}(\mathbf{q})\mathbf{v} + G^T(\mathbf{q})n(\mathbf{q},G(\mathbf{q})\mathbf{v}) = \mathbf{0}$$

Dynamic model of a unicycle

The dynamic model of the unicycle is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 1/m & 0 \\ 0 & 1/I \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$



where

- τ_1 is the driving force
- τ_2 is the steering torque

Do not use a cannon to kill a mosquito!

A few remarks on the Lagrange formulation

- it is an elegant and general tool to derive the dynamic model of a manipulator and a mobile robot
- it opens the way to a linearization law either for manipulators and mobile robots
- the dynamic model can be further extended in order to include actuator models
- another modelling tool, based on balances of forces and torques acting on the robot exists, as the Newton-Euler formulation for manipulators

Let’s derive the bicycle dynamic model using balances of forces and torques...

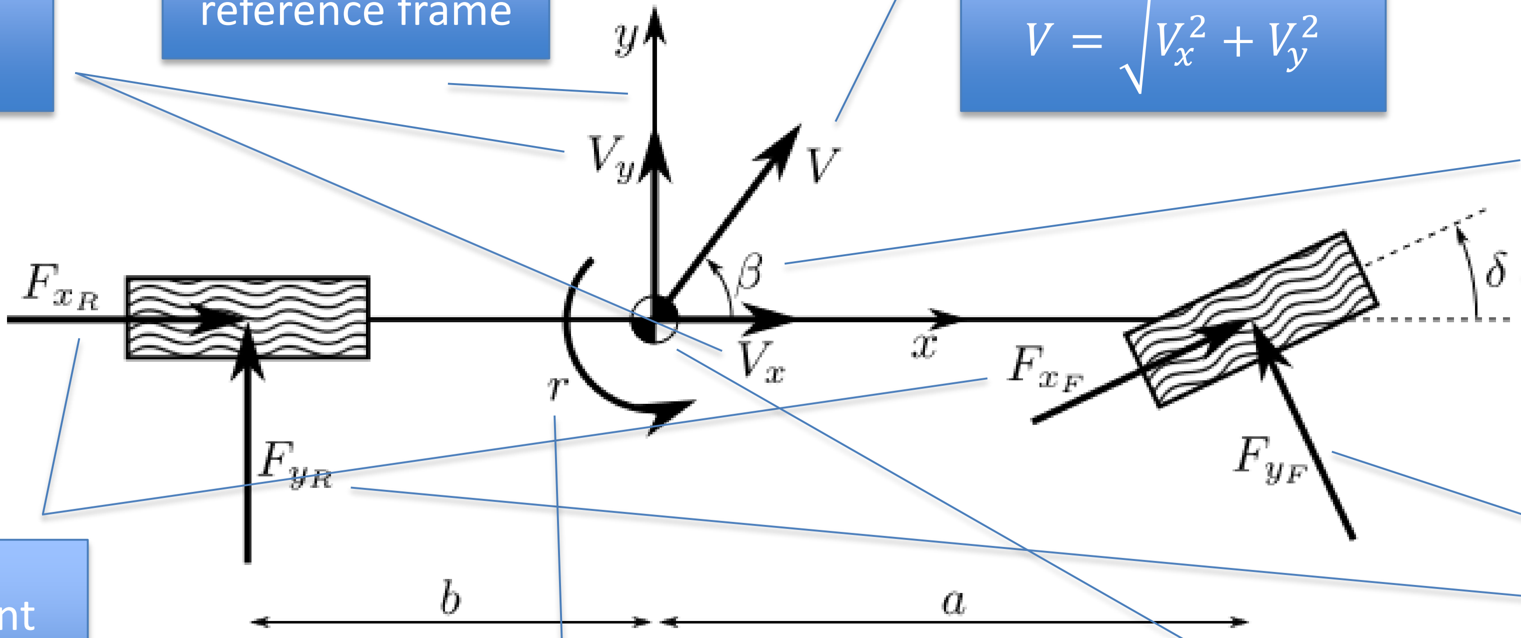
Dynamic model of a bicycle

Longitudinal (V_x) and lateral (V_y) velocities

Body-fixed reference frame

Absolute velocity
$V = \sqrt{V_x^2 + V_y^2}$

Sideslip angle
$\beta = \tan^{-1}\frac{V_y}{V_x}$



Steering angle

F_{x_R} and F_{x_F} represent traction / braking forces

Yaw rate
$r = \dot{\psi}$

Bicycle center of gravity

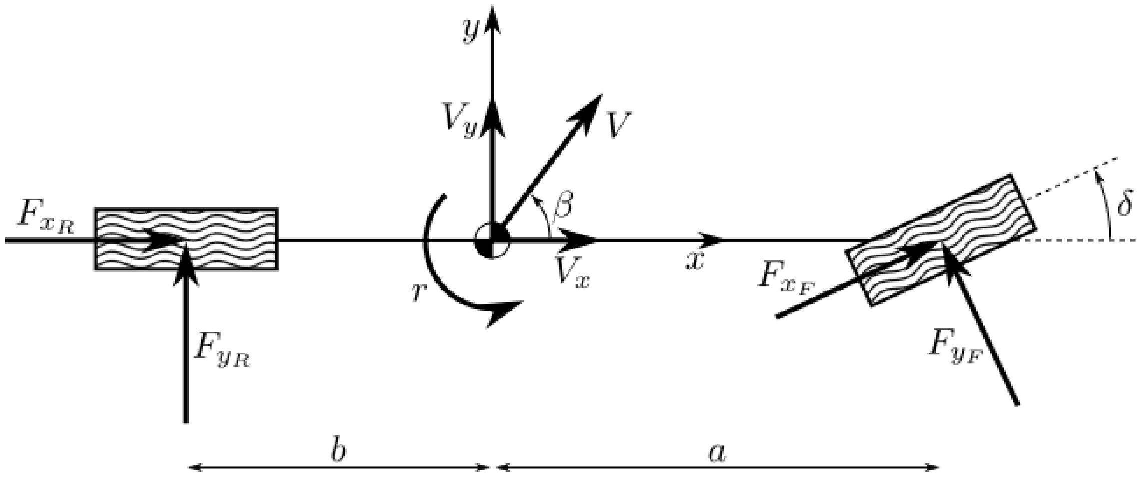
F_{y_R} and F_{y_F} represent tire-ground interaction forces

Dynamic model of a bicycle

Let's write the Newton equations for the translational motion of the center of mass in the x and y directions with respect to a *body-fixed reference frame*

$$F_{x_F}\cos(\delta) - F_{y_F}\sin(\delta) + F_{x_R} = Ma_x$$

$$F_{x_F}\sin(\delta) + F_{y_F}\cos(\delta) + F_{y_R} = Ma_y$$



And the Euler equation for the rotational motion (moments are referred to the center of mass)

$$a \cdot F_{x_F}\sin(\delta) + a \cdot F_{y_F}\cos(\delta) - b \cdot F_{y_R} = I_z\dot{r}$$

We have now to refer accelerations to an inertial frame.

Dynamic model of a bicycle

See additional material B for the derivation

The vehicle absolute velocity can be expressed as

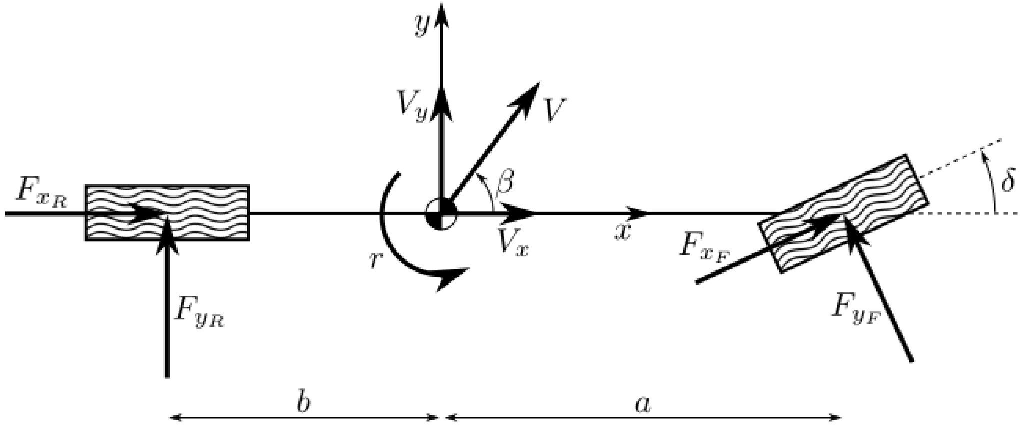
$$\vec{V} = V_x\vec{x} + V_y\vec{y}$$

and differentiating with respect to time we can refer accelerations to an inertial frame through relations

$$a_x = \dot{V}_x - rV_y \qquad a_y = \dot{V}_y + rV_x$$

We can thus express the model in the body-fixed reference frame as

$$F_{x_F}\cos(\delta) - F_{y_F}\sin(\delta) + F_{x_R} + MrV_y = M\dot{V}_x$$
$$F_{x_F}\sin(\delta) + F_{y_F}\cos(\delta) + F_{y_R} - MrV_x = M\dot{V}_y$$
$$a \cdot F_{x_F}\sin(\delta) + a \cdot F_{y_F}\cos(\delta) - b \cdot F_{y_R} = I_z\dot{r}$$



Dynamic model of a bicycle

The absolute velocity of the vehicle and the longitudinal and lateral velocities are related through the sideslip angle

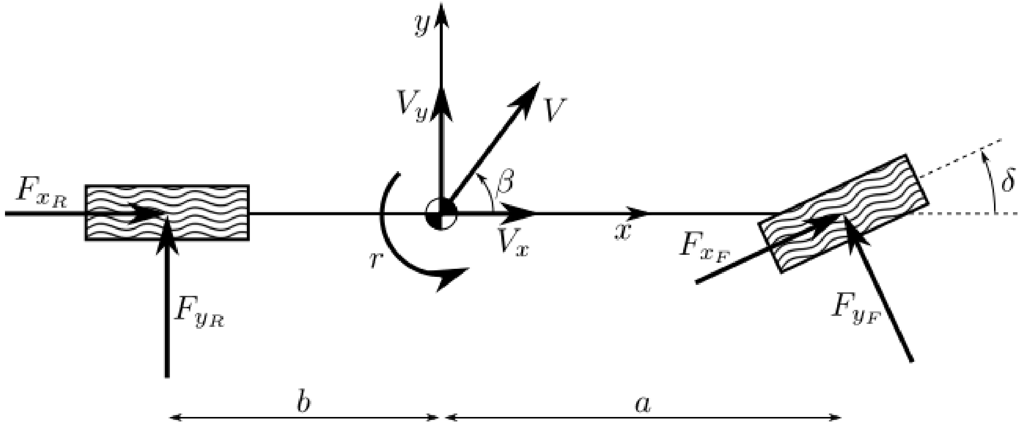
$$V_x = V\cos(\beta) \qquad V_y = V\sin(\beta)$$

and

$$\beta = \arctan\left(\frac{V_y}{V_x}\right)$$

As a consequence, the dynamic model can be expressed with respect to (V_x, V_y, r) or (V, β, r).

$$F_{x_F}\cos(\delta) - F_{y_F}\sin(\delta) + F_{x_R} + MrV_y = M\dot{V}_x$$
$$F_{x_F}\sin(\delta) + F_{y_F}\cos(\delta) + F_{y_R} - MrV_x = M\dot{V}_y$$
$$a\cdot F_{x_F}\sin(\delta) + a\cdot F_{y_F}\cos(\delta) - b\cdot F_{y_R} = I_z\dot{r}$$



See additional material C for the derivation

The two forms of the dynamic model are

$$M\dot{V}_x = F_{x_F}\cos(\delta) - F_{y_F}\sin(\delta) + F_{x_R} + MrV_y$$
$$M\dot{V}_y = F_{x_F}\sin(\delta) + F_{y_F}\cos(\delta) + F_{y_R} - MrV_x$$
$$I_z\dot{r} = a\cdot F_{x_F}\sin(\delta) + a\cdot F_{y_F}\cos(\delta) - b\cdot F_{y_R}$$

or

$$M\dot{V} = F_{x_F}\cos(\delta-\beta) - F_{y_F}\sin(\delta-\beta) + F_{x_R}\cos(\beta) + F_{y_R}\sin(\beta)$$
$$\dot{\beta} = \frac{1}{MV}\left(F_{x_F}\sin(\delta-\beta) + F_{y_F}\cos(\delta-\beta) + F_{y_R}\cos(\beta) - F_{x_R}\sin(\beta) - MrV\right)$$
$$I_z\dot{r} = a\cdot F_{x_F}\sin(\delta) + a\cdot F_{y_F}\cos(\delta) - b\cdot F_{y_R}$$

We need to include the kinematic states in both models.

The dynamic model already describes the behavior of the vehicle heading or yaw angle ψ, we have to add the x and y positions

$$\dot{x} = V\cos\left(\psi + \beta\right)$$
$$\dot{y} = V\sin\left(\psi + \beta\right)$$

Let's now analyze the dynamic model more in detail…

Dynamic model of a bicycle

We can introduce some simplifications in the bicycle dynamic model.

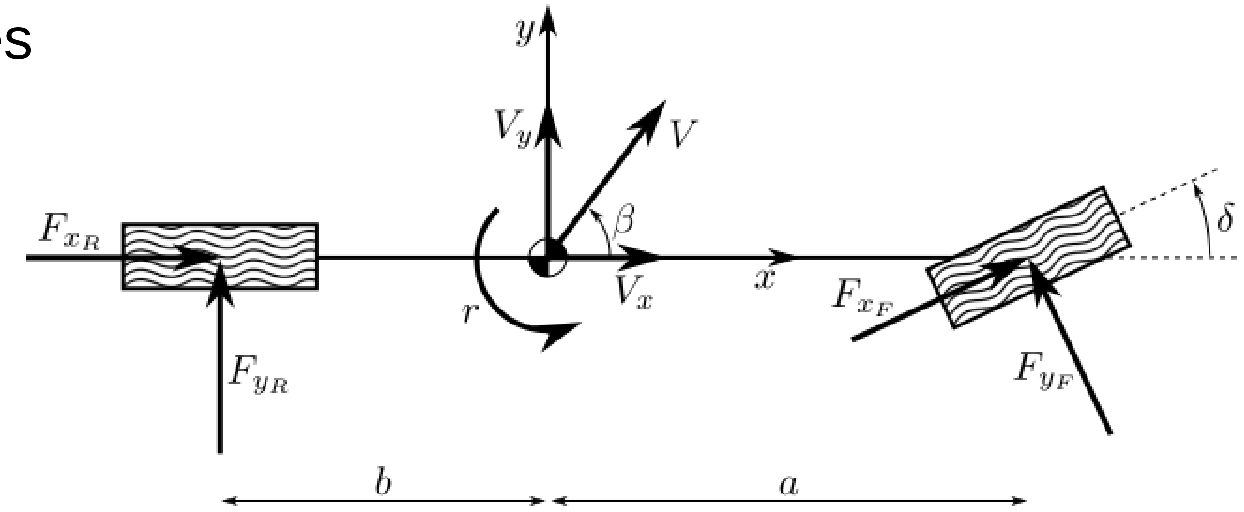
We have considered both F_{x_R} and F_{x_F}, they can represent

- a traction force
- a braking force

In mobile robotics braking can be sometimes neglected.

Assuming braking can be neglected and the bicycle is rear-wheel drive

$$F_{x_F} = 0$$



With the previous assumptions we obtain

$$M\dot{V}_x = -F_{y_F}\sin(\delta) + F_{x_R} + MrV_y$$
$$M\dot{V}_y = F_{y_F}\cos(\delta) + F_{y_R} - MrV_x$$
$$I_z\dot{r} = a\cdot F_{y_F}\cos(\delta) - b\cdot F_{y_R}$$

or, we the other set of states

$$M\dot{V} = -F_{y_F}\sin(\delta-\beta) + F_{x_R}\cos(\beta) + F_{y_R}\sin(\beta)$$
$$\dot{\beta} = \frac{1}{MV}\left(F_{y_F}\cos(\delta-\beta) + F_{y_R}\cos(\beta) - F_{x_R}\sin(\beta) - MrV\right)$$
$$I_z\dot{r} = a\cdot F_{y_F}\cos(\delta) - b\cdot F_{y_R}$$

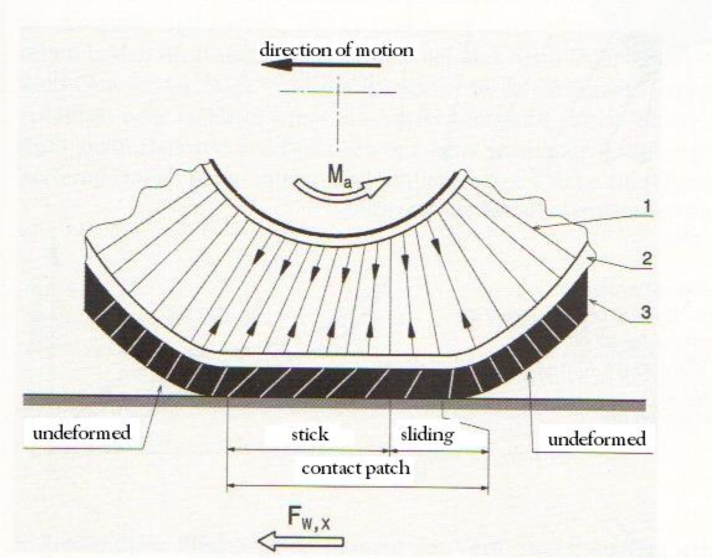
Further simplifications can be introduced considering that β is small.

The two modelling tools we have introduced allow to derive the dynamic model of the mobile robot up to the traction and steering forces.

In the case of a manipulator inputs are joint torques, they can be used as control variables, possibly providing a model of the actuators (electric motors at the joint).

In the case of a mobile robot inputs are longitudinal and lateral forces, they cannot be used as control variables as they are not generated by an actuator but by the interaction between tires and ground.

We conclude that the dynamic model have to be integrated with a model of the tire-ground interaction and, possibly, a model of the wheel actuator.



Let's start introducing a model of tires.

We introduce the following assumptions:

- road is an hard and flat surface (a geometric plane)
- tires are perfectly aligned (we neglect camber, caster and toe)

Tires are made from rubber, a material characterized by a nonlinear elastic structure with small hysteresis.

A portion of tire surface (*contact patch* or *footprint*) is in contact with road surface.

Tire deformation at the contact patch generate forces and torques that act on the vehicle.

We can represent all these forces and torques with a resultant force $\boldsymbol{F}$ and a resultant moment $\boldsymbol{M}$.

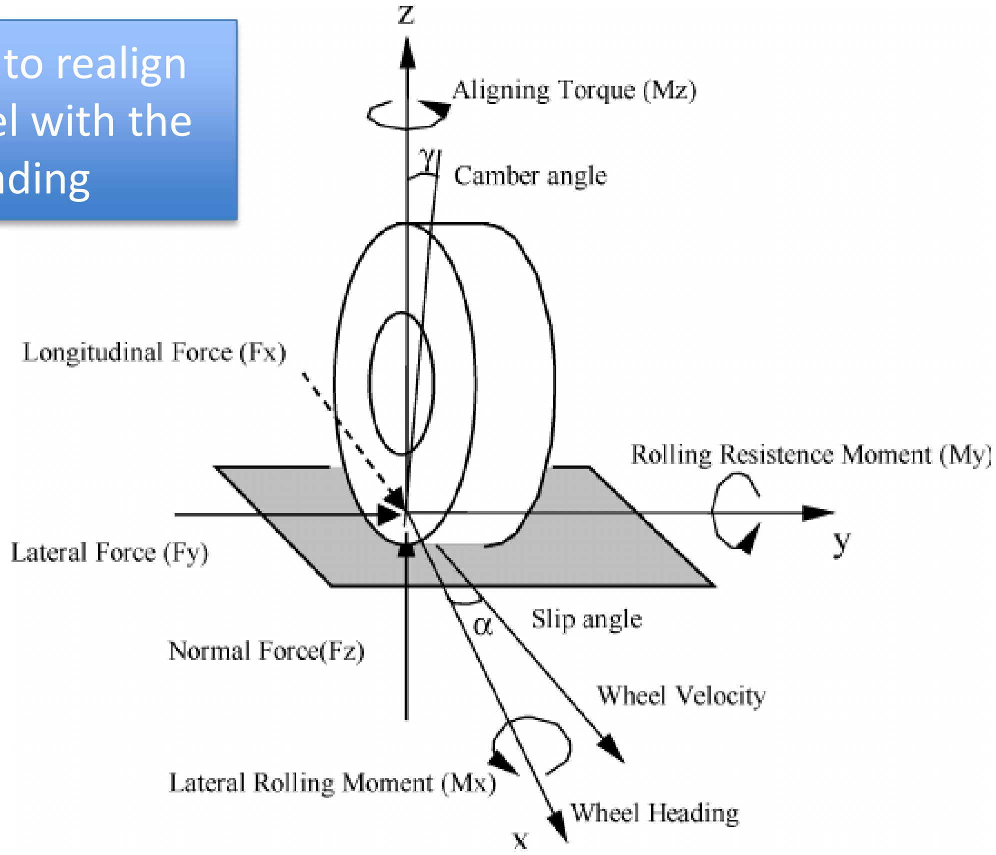
Tire modelling: tire-ground interaction

We can identify the following components in these resultants

- F_x, longitudinal force
- F_y, lateral force
- F_z, vertical load or normal force
- M_x, overturning moment
- M_y, rolling resistance moment
- M_z, self-aligning torque

A restoring moment to realign the direction of travel with the direction of heading

It is mainly caused by hysteresis in the tires (visco-elasticity of the material)



A wheel with tire under real operating conditions is usually not in pure rolling.

We thus introduce *tire slip indicators* saying how tire kinematical quantities are far from pure rolling conditions.

For example the *theoretical longitudinal slip* is defined as

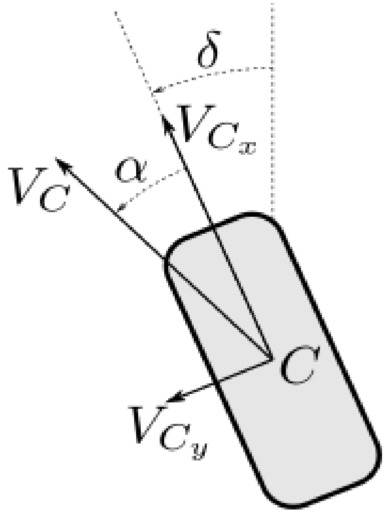
$$\sigma_x = \frac{V_x - r_e\Omega}{r_e\Omega}$$

If $r_e\Omega = 0$, locked brakes, $\sigma_x = \infty$, sliding without rotating

For the lateral motion, instead, we introduce the *slip angle* α

If $V_x = 0$ but $r_e\Omega \neq 0$, $\sigma_x = -1$, spinning with no motion

$$\tan(\alpha) = -\frac{V_{c_y}}{V_{c_x}}$$



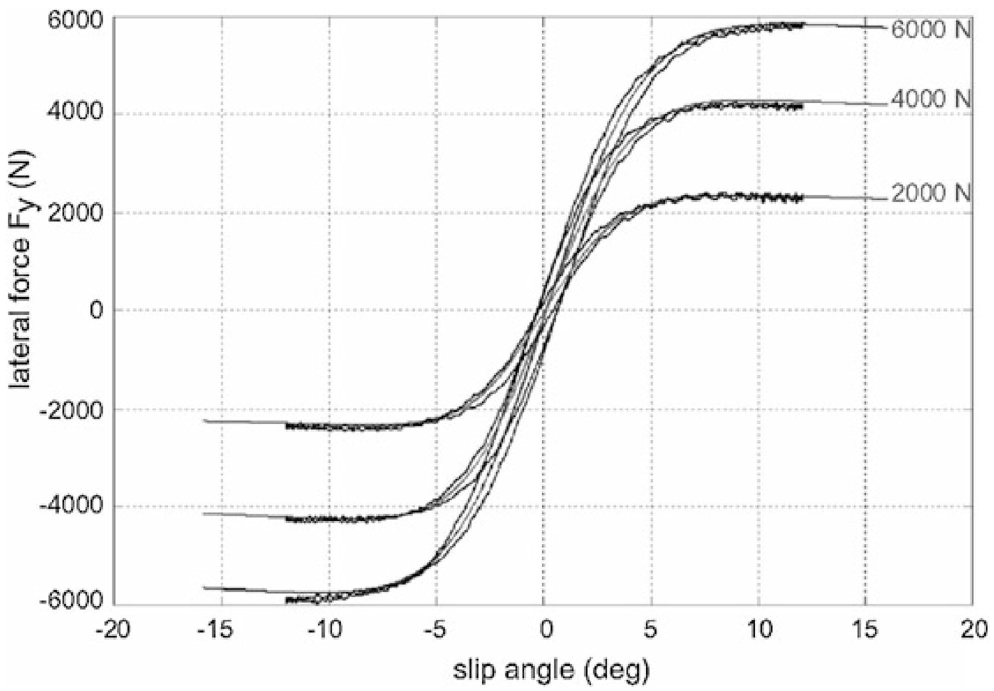
We can now characterize the tire behavior introducing a relation between resultant forces and moments and tire slips.

Tire modelling: the Magic Formula

We first introduce an *empirical tire model*.

Assuming that an experimental dataset including lateral forces and corresponding slip angles is available. These curves can be fit by a mathematical function in order to define a relation between force and slip.

The classical function used in tire modelling is the *Pacejka Magic Formula*



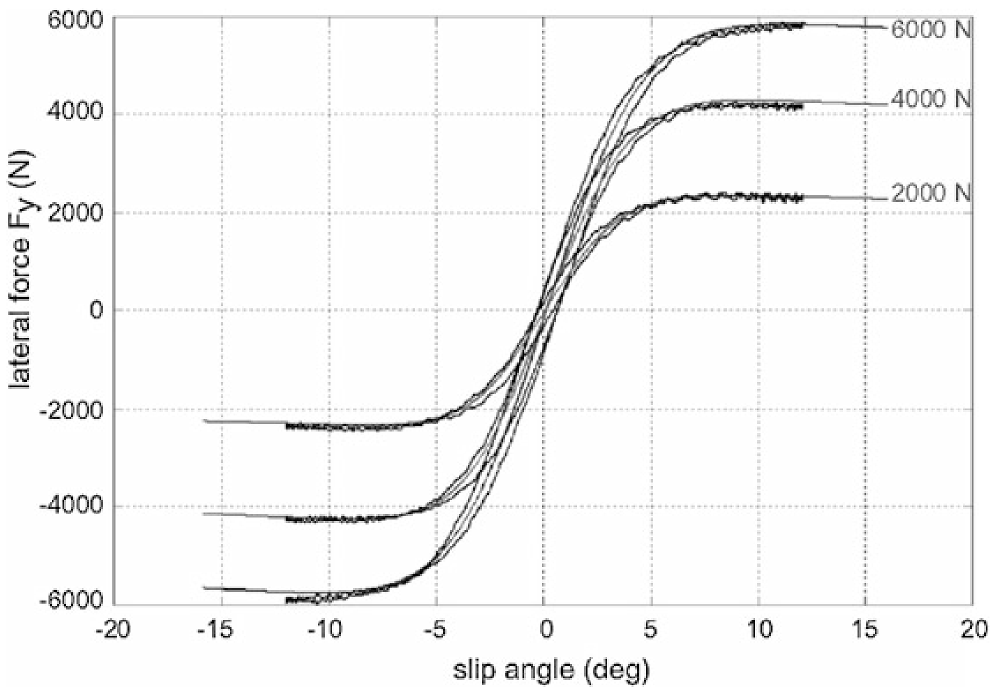
$$y(x) = D\sin\{C\arctan[Bx - E(Bx - \arctan(Bx))]\}$$

- $y(x)$: F_x or F_y with respect to the corresponding slip
- D: peak value
- C: shape factor
- B: stiffness value
- E: curvature factor

Tire modelling: the Magic Formula

The Magic Formula is characterized by many properties. We only observe that:

- curves depend on the vertical load, for this reason parameter D is usually assumed proportional to F_z
- the slope at the origin
  $$y'(0) = BCD$$
  is called *cornering stiffness* and it is particularly important as it allows to introduce a linear approximation of the curve (for low slip values)



A tire model based on the Magic Formula

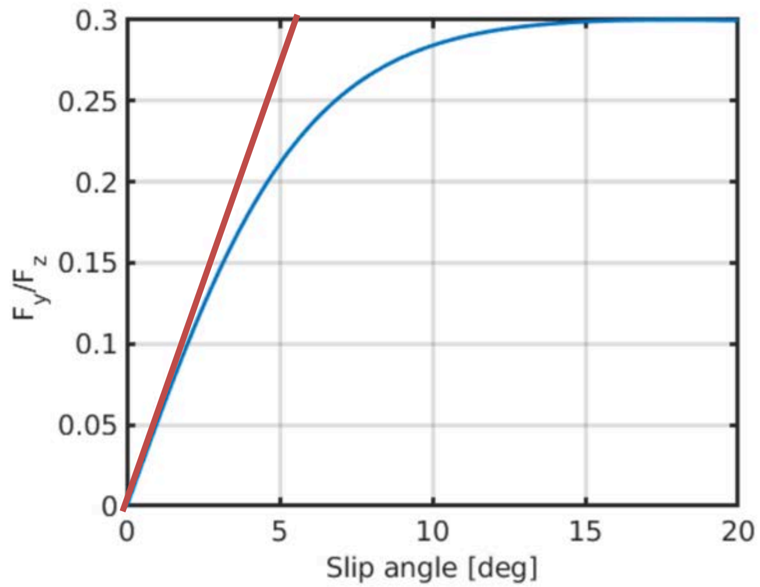
A tire lateral force has been characterized using the Pacejka Magic Formula

$$\frac{F_y}{F_z} = 0.3 \sin\left(2 \arctan\left(5\alpha - \left(5\alpha - \arctan\left(5\alpha\right)\right)\right)\right)$$

Determine the cornering stiffness and draw on the picture the cornering stiffness approximation.

The cornering stiffness referred to the normalized force F_y/F_z is given by BCD and it is equal to

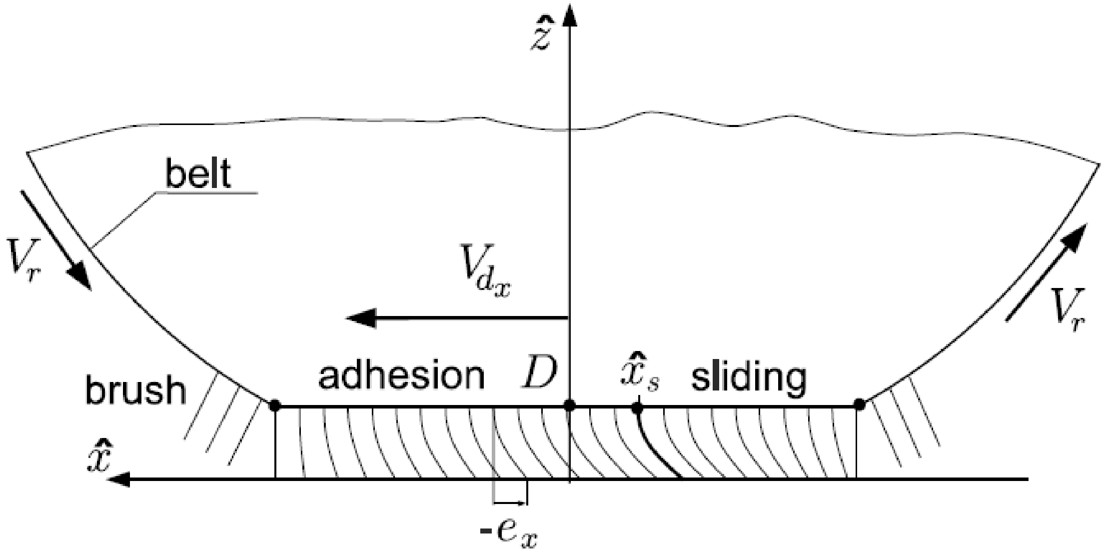
$$0.3 \cdot 2 \cdot 5 = 3\ \mathrm{Ns/rad} \approx 0.052\ \mathrm{Ns/deg}$$



Physical models can be introduced as well to characterize tire behavior.

The simplest physical model is the *brush model*. Though it is not as good as the Magic Formula in explaining experimental data, it is particularly interesting as it allows to understand the fundamental aspects of tire-road interaction.

We model the tire as a belt equipped with infinitely many flexible bristles (the thread), wrapped around a cylindrical rigid body (the rim), which moves on a flat surface (the roadway).



See additional material D for the derivation

Tire modelling: the brush model

The longitudinal force is given by

$$F_x = \begin{cases} C_x \sigma_x \left(-1 + \frac{|\sigma_x|}{\sigma_{x_{sl}}} - \frac{\sigma_x^2}{3\sigma_{x_{sl}}^2} \right) & |\sigma_x| < \sigma_{x_{sl}} \\ -\mu F_z \text{sign}(\sigma_x) & |\sigma_x| \geq \sigma_{x_{sl}} \end{cases}$$

Full adesion or adhesion/sliding

Full sliding

where C_x is the *longitudinal stiffness* of the tire, μ is the static friction coefficient, F_z the normal load, $\sigma_{x_{sl}}$ is the minimum slip value that gives full sliding.

The slip σ_x is

- positive, when the vehicle is braking
- negative, when the vehicle is accelerating

The lateral force is given by a similar expression

$$F_y = \begin{cases} C_\alpha z \left(-1 + \frac{|z|}{z_{sl}} - \frac{z^2}{3z_{sl}^2}\right) & |z| < z_{sl} \\ -\mu F_z \text{sign}(\alpha) & |z| \geq z_{sl} \end{cases}$$

where C_α is the *cornering stiffness* of the tire and $z = \tan\alpha$.

This model is also called *Fiala tire model*.

We have computed the longitudinal and lateral forces assuming the tire is having only lateral, or only longitudinal slip.

In general, we can have both slips at the same time. We must consider that the maximum tire force is limited by friction (*friction circle constraint*)

$$\sqrt{F_x^2 + F_y^2} \leq \mu F_z$$

Tire modelling: a comparison between Fiala and Pacejka models

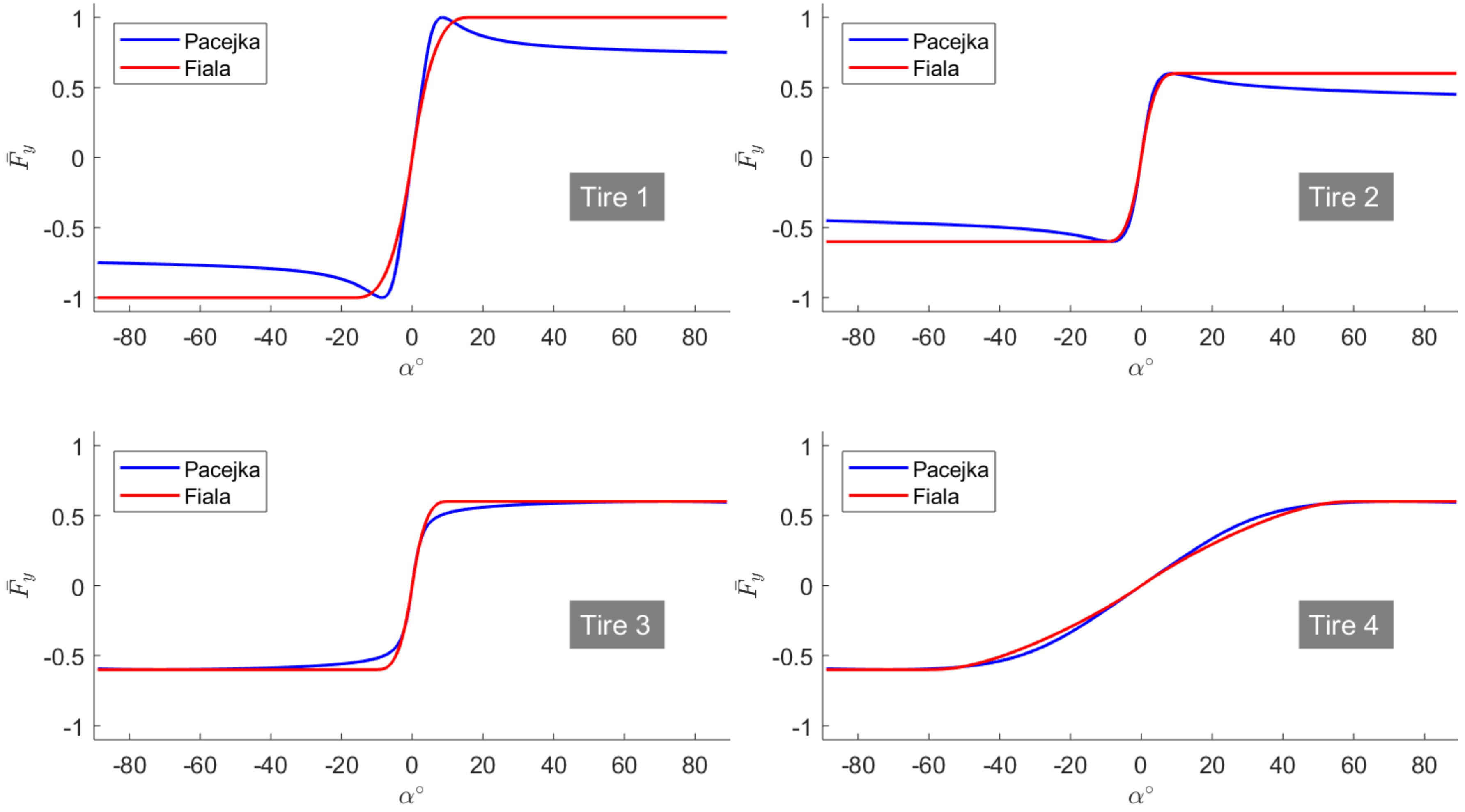
Pacejka model is more complex, but better fits experimental data

Fiala model is simpler, but cannot represent the peak

As a consequence...

... Pacejka model is good for accurate simulation

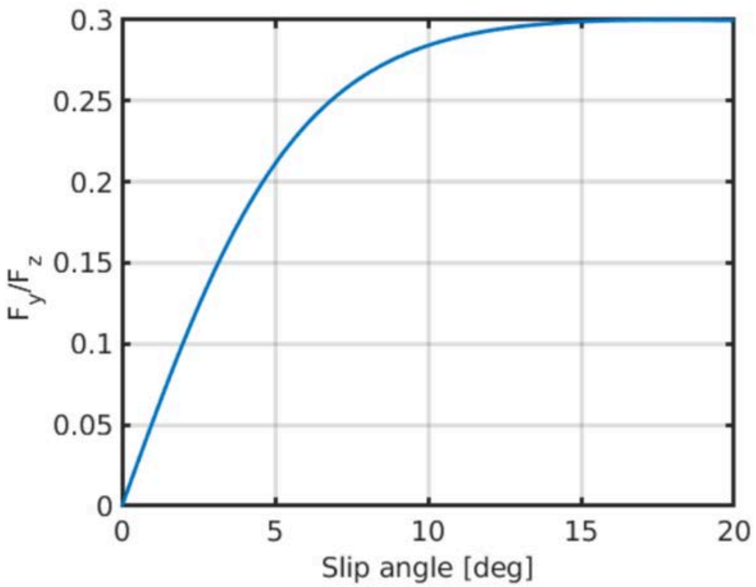
... Fiala model is good for model-based control design



Maximum tire force and friction circle constraint

Consider again the same tire, whose lateral force is characterized by the following Pacejka Magic Formula

$$\frac{F_y}{F_z} = 0.3\sin\left(2\arctan\left(5\alpha - \left(5\alpha - \arctan\left(5\alpha\right)\right)\right)\right)$$



The characterization is based on a laboratory experiment where $F_x = 0$ and only F_y is considered.

During a curve in a field experiment the tire is characterised by a slip angle of $5\ deg$.

What is the corresponding value of the lateral force F_y, assuming $F_z = 150\ N$?

What is the maximum longitudinal force F_x the tire can generate in these conditions?

Maximum tire force and friction circle constraint

The lateral force is given by

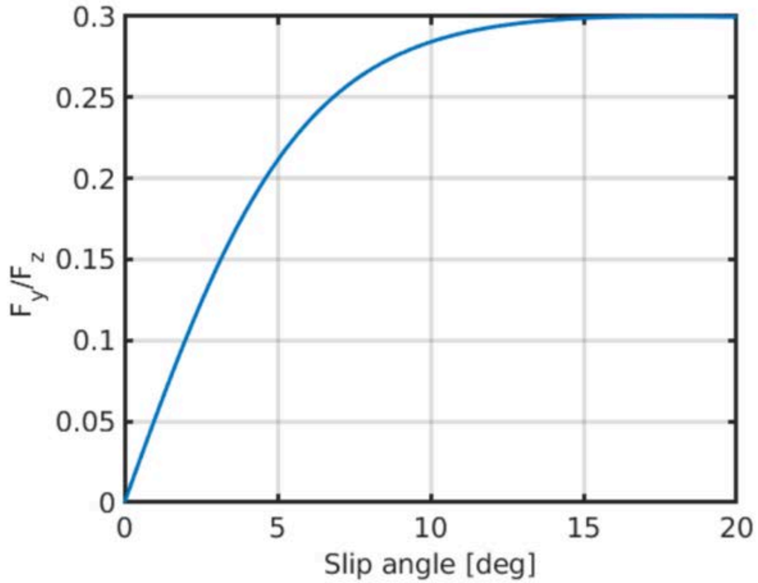
$$F_y = 150 \cdot 0.3 \sin\left(2 \arctan\left(5 \frac{5\pi}{180} - \left(5 \frac{5\pi}{180} - \arctan\left(5 \frac{5\pi}{180}\right)\right)\right)\right) = 31.67\ \mathrm{N}$$

From the picture it follows that the maximum value of F_y/F_z is 0.3, and thus $\mu = 0.3$.

The maximum force the tire can generate is
$150 \cdot 0.3 = 45\ N$.

According to the friction circle constraint, the maximum value of the longitudinal force F_x is given by

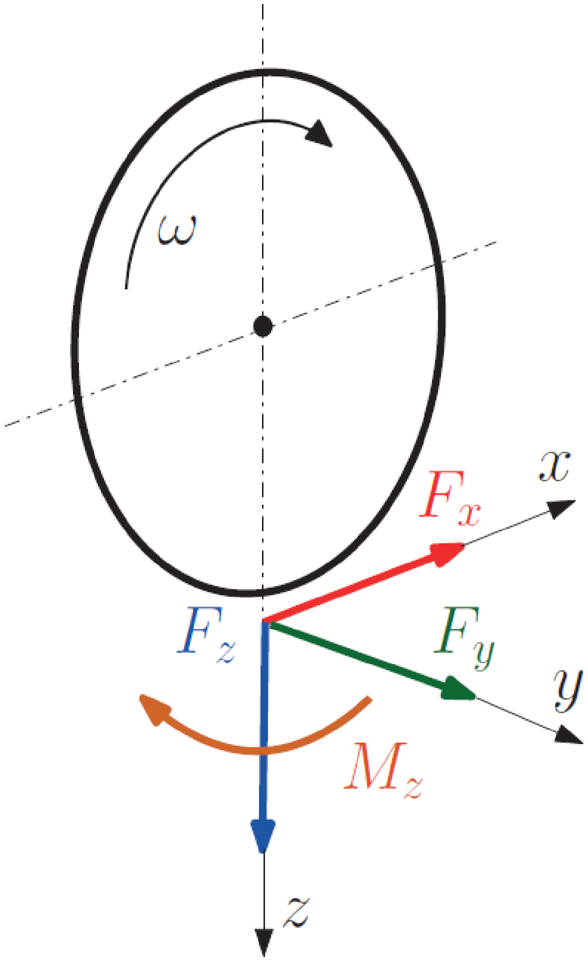
$$F_x = \sqrt{\mu^2 F_z^2 - F_y^2} = 31.97\ \mathrm{N}$$



Besides considering the tire-ground interaction, we should also take into account:

- wheel dynamics
- actuator dynamics

They are however definitely faster with respect to the other considered dynamics, we can thus neglect them.



We start again from the bicycle model we have already introduced

$$M a_x = F_{x_F} \cos(\delta) - F_{y_F} \sin(\delta) + F_{x_R}$$
$$M a_y = F_{x_F} \sin(\delta) + F_{y_F} \cos(\delta) + F_{y_R}$$
$$I_z \dot{r} = a\left(F_{x_F} \sin(\delta) + F_{y_F} \cos(\delta)\right) - b F_{y_R}$$

if we assume small values for β ($\cos\beta \approx 1$, $\sin\beta \approx \beta$) the motion model becomes

$$M\dot{V} = F_{x_F} \cos(\delta) - F_{y_F} \sin(\delta) + F_{x_R}$$
$$MV\left(\dot{\beta} + r\right) = F_{x_F} \sin(\delta) + F_{y_F} \cos(\delta) + F_{y_R}$$
$$I_z \dot{r} = a\left(F_{x_F} \sin(\delta) + F_{y_F} \cos(\delta)\right) - b F_{y_R}$$

Assuming also small values for δ ($\cos\delta \approx 1$, $\sin\delta \approx \delta$) we obtain…

See additional material E for the derivation

Assuming also small values for δ ($\cos\delta \approx 1$, $\sin\delta \approx \delta$) we obtain

$$M\dot{V} = F_{x_F} + F_{x_R}$$

$$MV\left(\dot{\beta} + r\right) = F_{y_F} + F_{y_R}$$

$$I_z\dot{r} = aF_{y_F} - bF_{y_R}$$

For the lateral forces F_{y_F} and F_{y_R} we assume a linear relation

$$F_{y_F} = -C_{\alpha_F}\alpha_F$$

$$F_{y_R} = -C_{\alpha_R}\alpha_R$$

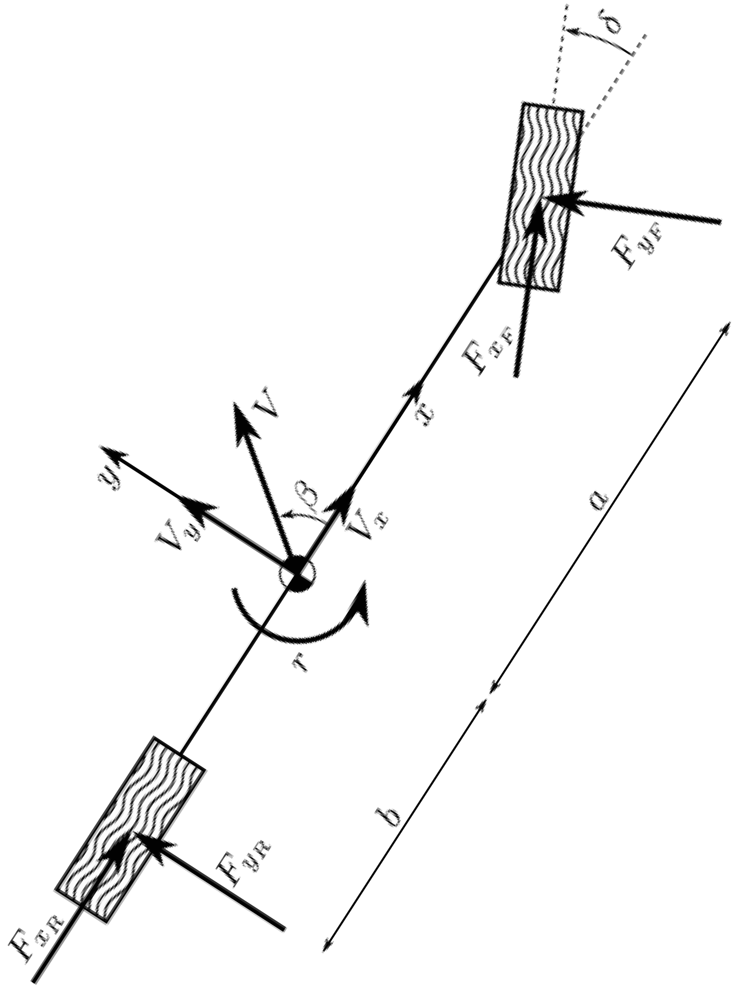
To compute the tire slip angles we need the expression of the front and rear wheel velocities

$$V_{x_F} = V\cos(\beta) \approx V$$

$$V_{y_F} = V\sin(\beta) + ar \approx V\beta + ar$$

$$V_{x_R} = V\cos(\beta) \approx V$$

$$V_{y_R} = V\sin(\beta) - br \approx V\beta - br$$



The slip angles are given by

$$\alpha_F = \arctan\left(\frac{V_{y_F}}{V_{x_F}}\right) - \delta \approx \frac{V_{y_F}}{V_{x_F}} - \delta \approx \frac{V\beta + ar}{V} - \delta = \beta + \frac{ar}{V} - \delta$$

$$\alpha_R = \arctan\left(\frac{V_{y_R}}{V_{x_R}}\right) \approx \frac{V_{y_R}}{V_{x_R}} \approx \frac{V\beta - br}{V} = \beta - \frac{br}{V}$$

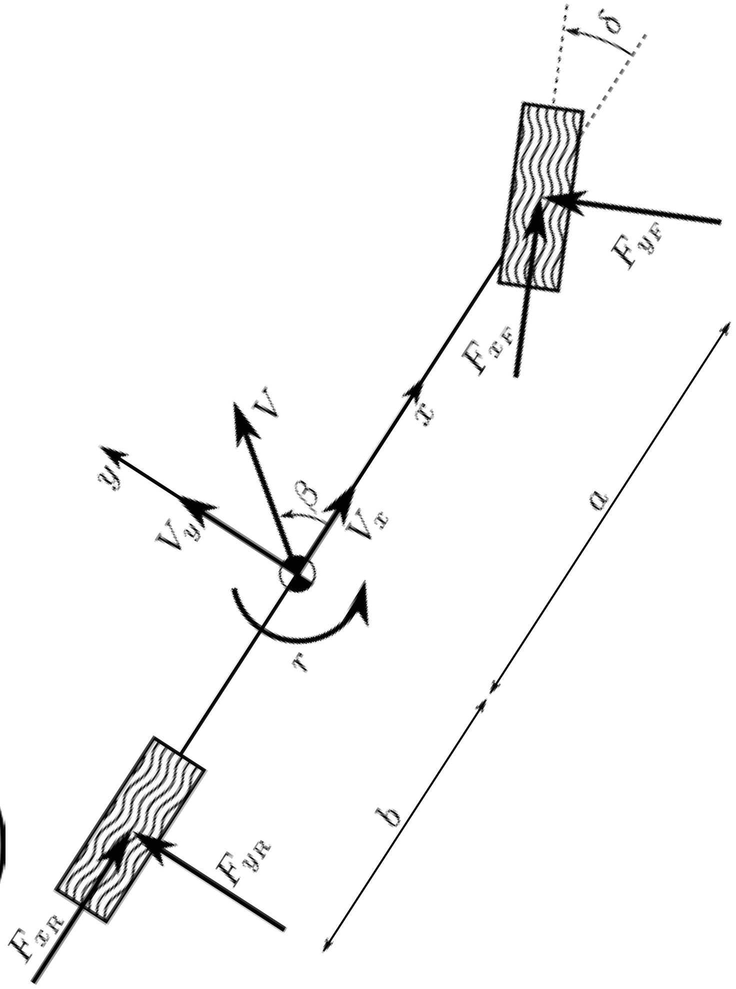
If we consider V slowly varying, the first equation

$$M\dot{V} = F_{x_F} + F_{x_R}$$

can be considered at steady state, and thus neglected.

From the second equation, instead, we obtain

$$MV\left(\dot{\beta} + r\right) = -C_{\alpha_F}\alpha_F - C_{\alpha_R}\alpha_R = -C_{\alpha_F}\left(\beta + \frac{ar}{V} - \delta\right) - C_{\alpha_R}\left(\beta - \frac{br}{V}\right)$$



From the second equation we thus obtain the sideslip dynamics

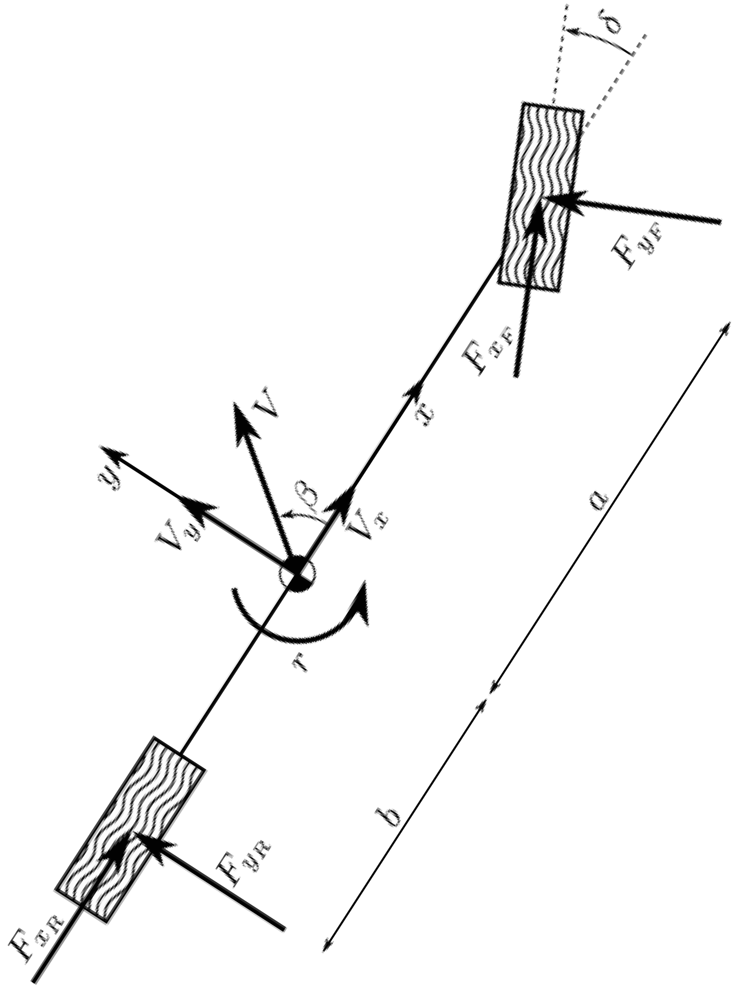
$$\dot{\beta} = -\frac{C_{\alpha_F} + C_{\alpha_R}}{MV}\beta + \left(\frac{bC_{\alpha_R} - aC_{\alpha_F}}{MV^2} - 1\right) r + \frac{C_{\alpha_F}}{MV}\delta$$

Finally, from the third equation we obtain

$$I_z \dot{r} = -aC_{\alpha_F}\alpha_F + bC_{\alpha_R}\alpha_R = -aC_{\alpha_F}\left(\beta + \frac{ar}{V} - \delta\right) + bC_{\alpha_R}\left(\beta - \frac{br}{V}\right)$$

The yaw dynamics is thus

$$\dot{r} = \frac{bC_{\alpha_R} - aC_{\alpha_F}}{I_z}\beta - \frac{a^2 C_{\alpha_F} + b^2 C_{\alpha_R}}{VI_z} r + \frac{aC_{\alpha_F}}{I_z}\delta$$

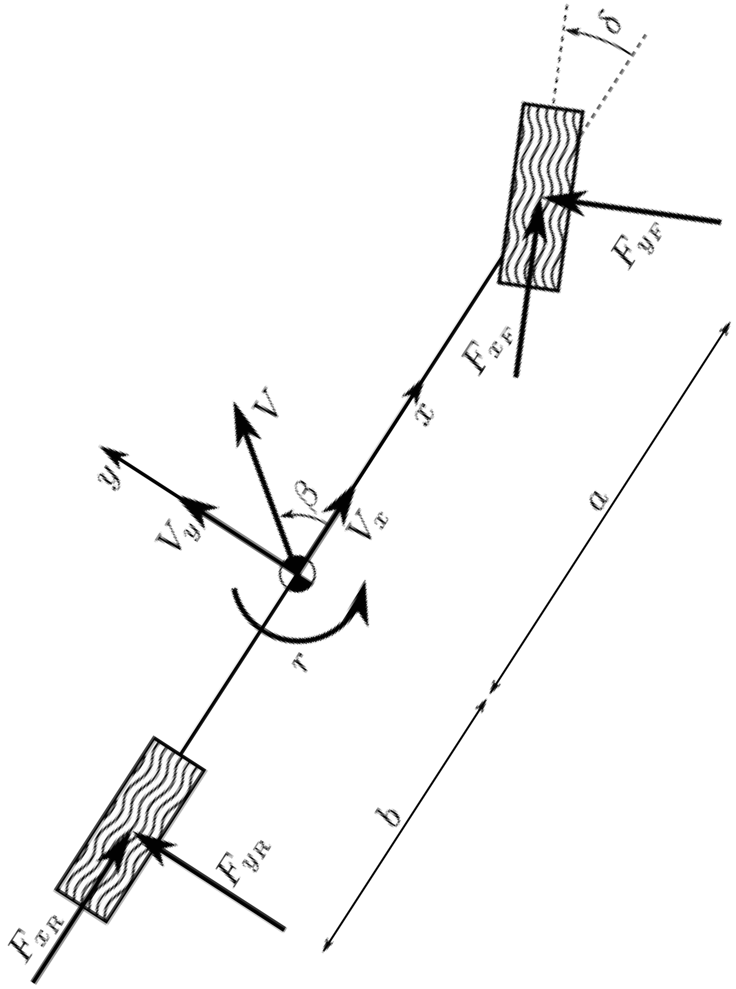


Simplified dynamic model of a bicycle

Putting all the equations together we obtain

$$\dot{\beta} = -\frac{C_{\alpha_F} + C_{\alpha_R}}{MV}\beta + \left(\frac{bC_{\alpha_R} - aC_{\alpha_F}}{MV^2} - 1\right) r + \frac{C_{\alpha_F}}{MV}\delta$$

$$\dot{r} = \frac{bC_{\alpha_R} - aC_{\alpha_F}}{I_z}\beta - \frac{a^2 C_{\alpha_F} + b^2 C_{\alpha_R}}{VI_z} r + \frac{aC_{\alpha_F}}{I_z}\delta$$



Though we have introduced simplifying assumptions, assuming small sideslip and small steering angles do not cause significant errors if we consider

- driving not at the limit of handling
- steering angles up to 45°

Conclusion

Given any mobile robot (ground, underwater/surface, aerial), the dynamic model can be derived using the Lagrange or the Newton-Euler approach.

The Lagrange approach, though more elegant, is rather complex, but it gives rise to a general model that allows to introduce a general linearization law.

The Newton-Euler approach is more simple, especially for simple systems like the unicycle and the bicycle.

The unicycle and bicycle dynamic models allow to represent many different common vehicles and mobile robots.

Dynamic model of a mobile robot: computing the derivative of *G(q)*

Let's start again from

$$\dot{\mathbf{q}} = G(\mathbf{q})\,\mathbf{v}$$

that is equivalent to

$$\dot{q}_1 = \sum_{j=1}^{m} g_{1j}(\mathbf{q})\,v_j \qquad \cdots \qquad \dot{q}_n = \sum_{j=1}^{m} g_{nj}(\mathbf{q})\,v_j$$

differentiating with respect to time we obtain

$$\ddot{q}_1 = \sum_{j=1}^{m} \left(\frac{\partial g_{1j}(\mathbf{q})}{\partial \mathbf{q}} \dot{\mathbf{q}} \right) v_j + \sum_{j=1}^{m} g_{1j}(\mathbf{q})\,\dot{v}_j \qquad \cdots \qquad \ddot{q}_n = \sum_{j=1}^{m} \left(\frac{\partial g_{nj}(\mathbf{q})}{\partial \mathbf{q}} \dot{\mathbf{q}} \right) v_j + \sum_{j=1}^{m} g_{nj}(\mathbf{q})\,\dot{v}_j$$

Dynamic model of a mobile robot : computing the derivative of *G(q)*

As a consequence we have

$$\dot{G}(\mathbf{q})\,\mathbf{v} = \begin{bmatrix} \sum_{j=1}^{m}\left(\frac{\partial g_{1j}(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)v_j \\ \vdots \\ \sum_{j=1}^{m}\left(\frac{\partial g_{nj}(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}}\right)v_j \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{m}\left(v_j\frac{\partial g_{1j}(\mathbf{q})}{\partial \mathbf{q}}\right)\dot{\mathbf{q}} \\ \vdots \\ \sum_{j=1}^{m}\left(v_j\frac{\partial g_{nj}(\mathbf{q})}{\partial \mathbf{q}}\right)\dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{m}\left(v_j\frac{\partial g_{1j}(\mathbf{q})}{\partial \mathbf{q}}\right) \\ \vdots \\ \sum_{j=1}^{m}\left(v_j\frac{\partial g_{nj}(\mathbf{q})}{\partial \mathbf{q}}\right) \end{bmatrix}\dot{\mathbf{q}}$$

that can be rewritten as

Dynamic model of a mobile robot: computing the derivative of *G(q)*

$$\dot{G}(\mathbf{q})\,\mathbf{v} = \begin{bmatrix} \sum_{j=1}^{m}\left(v_j \frac{\partial g_{1j}(\mathbf{q})}{\partial \mathbf{q}}\right) \\ \vdots \\ \sum_{j=1}^{m}\left(v_j \frac{\partial g_{nj}(\mathbf{q})}{\partial \mathbf{q}}\right) \end{bmatrix} \dot{\mathbf{q}} = \begin{bmatrix} v_1 \frac{\partial g_{11}(\mathbf{q})}{\partial \mathbf{q}} + v_2 \frac{\partial g_{12}(\mathbf{q})}{\partial \mathbf{q}} + \cdots + v_m \frac{\partial g_{1m}(\mathbf{q})}{\partial \mathbf{q}} \\ \vdots \\ v_1 \frac{\partial g_{n1}(\mathbf{q})}{\partial \mathbf{q}} + v_2 \frac{\partial g_{n2}(\mathbf{q})}{\partial \mathbf{q}} + \cdots + v_m \frac{\partial g_{nm}(\mathbf{q})}{\partial \mathbf{q}} \end{bmatrix} \dot{\mathbf{q}}$$

or

$$\dot{G}(\mathbf{q})\,\mathbf{v} = \left\{ v_1 \begin{bmatrix} \frac{\partial g_{11}(\mathbf{q})}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial g_{n1}(\mathbf{q})}{\partial \mathbf{q}} \end{bmatrix} + v_2 \begin{bmatrix} \frac{\partial g_{12}(\mathbf{q})}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial g_{n2}(\mathbf{q})}{\partial \mathbf{q}} \end{bmatrix} + \cdots + v_m \begin{bmatrix} \frac{\partial g_{1m}(\mathbf{q})}{\partial \mathbf{q}} \\ \vdots \\ \frac{\partial g_{nm}(\mathbf{q})}{\partial \mathbf{q}} \end{bmatrix} \right\} \dot{\mathbf{q}}$$

$$= \sum_{i=1}^{m}\left(v_i \frac{\partial \mathbf{g}_i(\mathbf{q})}{\partial \mathbf{q}}\right)\dot{\mathbf{q}} = \sum_{i=1}^{m}\left(v_i \frac{\partial \mathbf{g}_i(\mathbf{q})}{\partial \mathbf{q}}\right) G(\mathbf{q})\,\mathbf{v}$$

The vehicle absolute velocity can be expressed as

$$\vec{V} = V_x\vec{x} + V_y\vec{y}$$

and differentiating with respect to time

$$\frac{d\vec{V}}{dt} = a_x\vec{x} + a_y\vec{y} = \dot{V}_x\vec{x} + \dot{V}_y\vec{y} + V_x\frac{d\vec{x}}{dt} + V_y\frac{d\vec{y}}{dt} = \dot{V}_x\vec{x} + \dot{V}_y\vec{y} + \vec{r}\times\vec{V}$$

$$= \dot{V}_x\vec{x} + \dot{V}_y\vec{y} + \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ 0 & 0 & r \\ V_x & V_y & 0 \end{vmatrix} = \left(\dot{V}_x - rV_y\right)\vec{x} + \left(\dot{V}_y + rV_x\right)\vec{y}$$

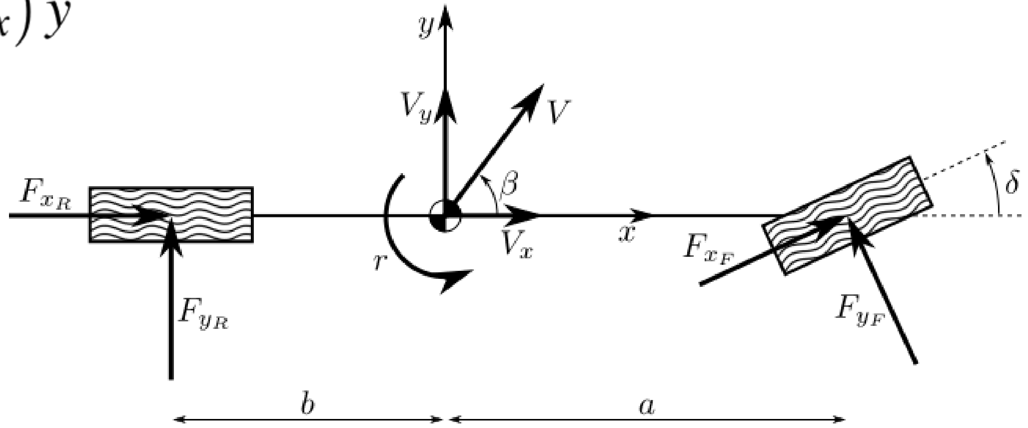
$\vec{r}\times\vec{x}$

$\vec{r}\times\vec{y}$

Here Poisson theorem is used

We can thus refer accelerations to an inertial frame through relations

$$a_x = \dot{V}_x - rV_y \qquad a_y = \dot{V}_y + rV_x$$



Dynamic model of a bicycle: writing the model in the (V, β, r) form

From

$$V = \sqrt{V_x^2 + V_y^2} \qquad \Rightarrow \qquad V^2 = V_x^2 + V_y^2$$

differentiating with respect to time

$$2V\dot{V} = 2V_x\dot{V}_x + 2V_y\dot{V}_y \qquad \Rightarrow \qquad \dot{V} = \dot{V}_x \cos(\beta) + \dot{V}_y \sin(\beta)$$

Substituting the previous equations we obtain

$$M\dot{V} = F_{x_F} \cos(\delta)\cos(\beta) - F_{y_F} \sin(\delta)\cos(\beta) + F_{x_R} \cos(\beta) + MrV_y \cos(\beta) +$$
$$F_{x_F} \sin(\delta)\sin(\beta) + F_{y_F} \cos(\delta)\sin(\beta) + F_{y_R} \sin(\beta) - MrV_x \sin(\beta)$$

and using the angle sum formulae

$$M\dot{V} = F_{x_F} \cos(\delta - \beta) - F_{y_F} \sin(\delta - \beta) + F_{x_R} \cos(\beta) + F_{y_R} \sin(\beta) + Mr(V_y \cos(\beta) - V_x \sin(\beta))$$

Dynamic model of a bicycle: writing the model in the (V, β, r) form

Observing that

$$Mr\left(V_y \cos(\beta) - V_x \sin(\beta)\right) = MrV\left(\sin(\beta)\cos(\beta) - \cos(\beta)\sin(\beta)\right) = 0$$

we conclude

$$M\dot{V} = F_{x_F}\cos(\delta - \beta) - F_{y_F}\sin(\delta - \beta) + F_{x_R}\cos(\beta) + F_{y_R}\sin(\beta)$$

The equation on sideslip can be derived from its definition

$$\tan(\beta) = \frac{V_y}{V_x}$$

differentiating with respect to time and considering that β is small

$$\left(1 + \tan^2(\beta)\right)\dot{\beta} = \frac{\dot{V}_y V_x - \dot{V}_x V_y}{V_x^2} \qquad \Rightarrow \qquad \dot{\beta} = \frac{\dot{V}_y \cos(\beta) - \dot{V}_x \sin(\beta)}{V}$$

Dynamic model of a bicycle: writing the model in the (V, β, r) form

Substituting the equations on velocities we obtain

$$\dot{\beta} = \frac{1}{MV}\left(F_{x_F}\sin(\delta)\cos(\beta) + F_{y_F}\cos(\delta)\cos(\beta) + F_{y_R}\cos(\beta) - MrV_x\cos(\beta)\right.$$
$$\left. - F_{x_F}\cos(\delta)\sin(\beta) + F_{y_F}\sin(\delta)\sin(\beta) - F_{x_R}\sin(\beta) - MrV_y\sin(\beta)\right)$$

and using the angle sum formulae

$$\dot{\beta} = \frac{1}{MV}\left(F_{x_F}\sin(\delta-\beta) + F_{y_F}\cos(\delta-\beta) + F_{y_R}\cos(\beta) - F_{x_R}\sin(\beta) - Mr\left(V_x\cos(\beta) + V_y\sin(\beta)\right)\right)$$

Observing that

$$V_x\cos(\beta) + V_y\sin(\beta) = V\cos^2(\beta) + V\sin^2(\beta) = V$$

we conclude

$$\dot{\beta} = \frac{1}{MV}\left(F_{x_F}\sin(\delta-\beta) + F_{y_F}\cos(\delta-\beta) + F_{y_R}\cos(\beta) - F_{x_R}\sin(\beta) - MrV\right)$$

Tire modelling: brush model derivation

Let's introduce two fundamental velocities

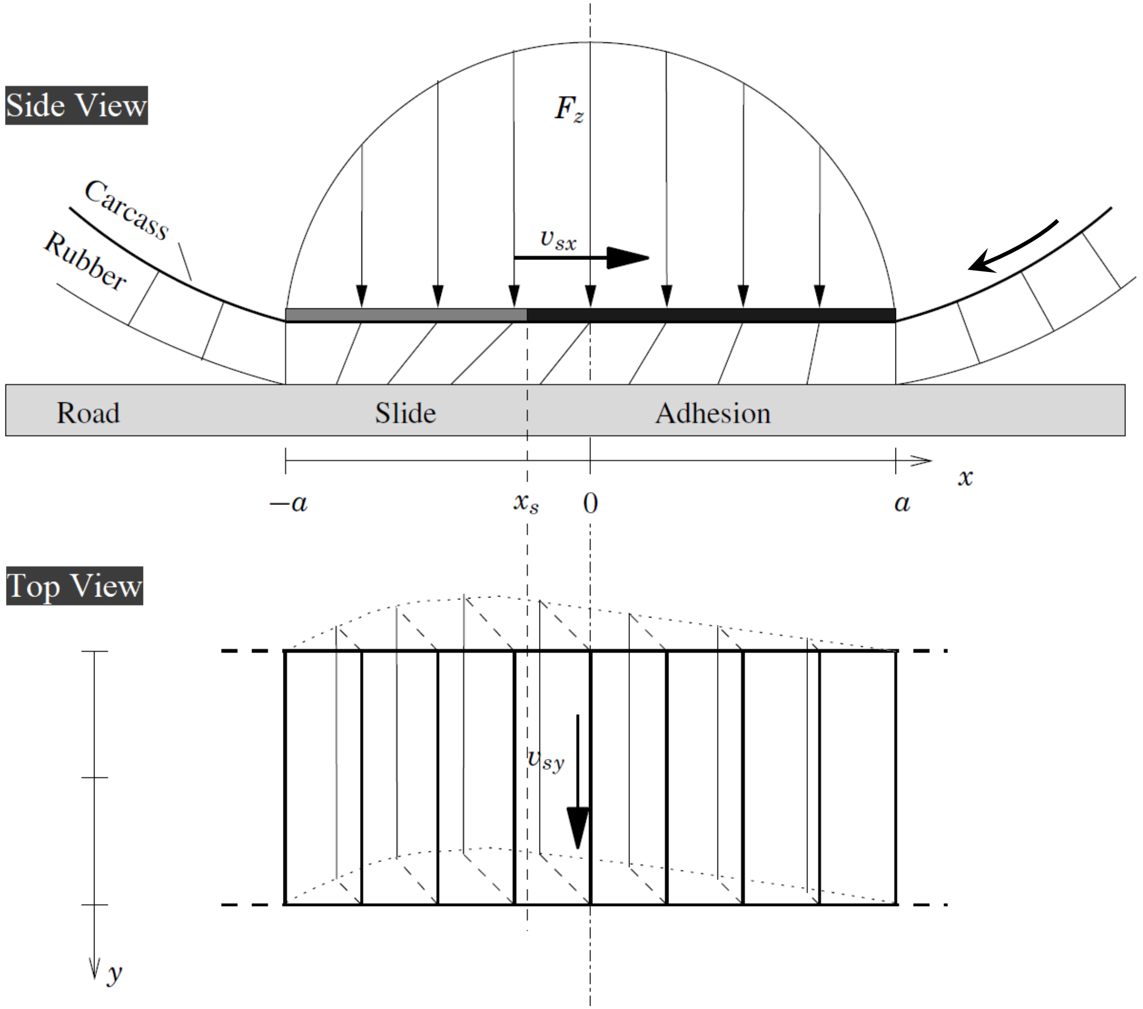
- V_x longitudinal velocity of the tire
- V_{sx} velocity of the wheel body at the carcass border relative to the road

whose ratio represents the ***practical longitudinal wheel sleep***

$$\kappa = \frac{V_{sx}}{V_x} = \frac{V_x - \Omega r_e}{V_x}$$

In a similar way we can define the ***practical tire slip angle***

$$\alpha = \arctan\left(\frac{V_{sy}}{V_x}\right)$$



Let's try to compute the force that deforms the bristles.

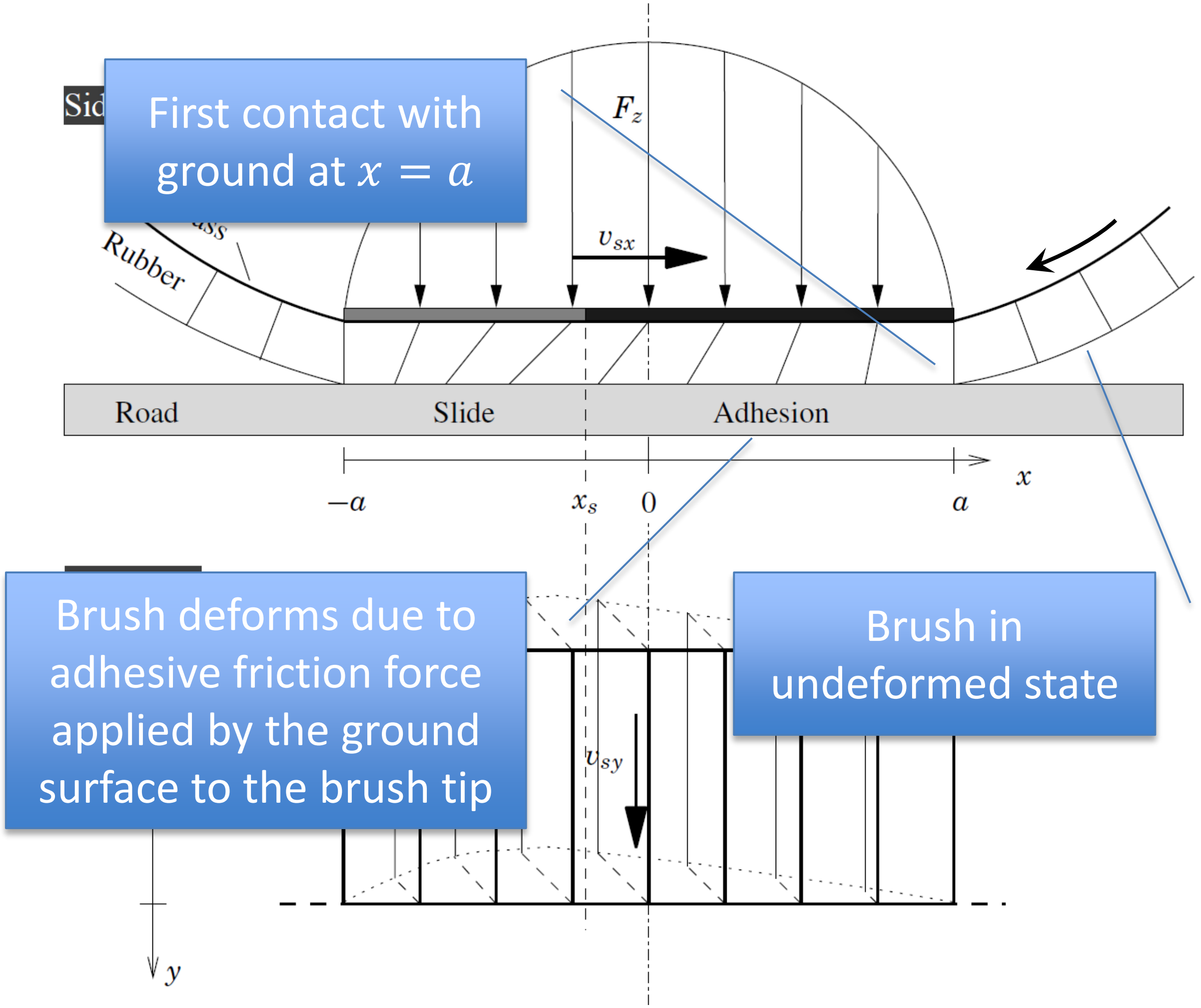
We consider a rectangular contact patch of length $2a$.

In the adhesion region, the position of the i-th brush upper point (attached to the carcass) is given by

$$x_{ci} = a - \int_0^t \Omega r_e \mathrm{d}t$$

and the brush lower point in contact with ground

$$x_{ri} = a - \int_0^t V_x \mathrm{d}t$$



Tire modelling: brush model derivation

The deformation of the i-th brush is given by

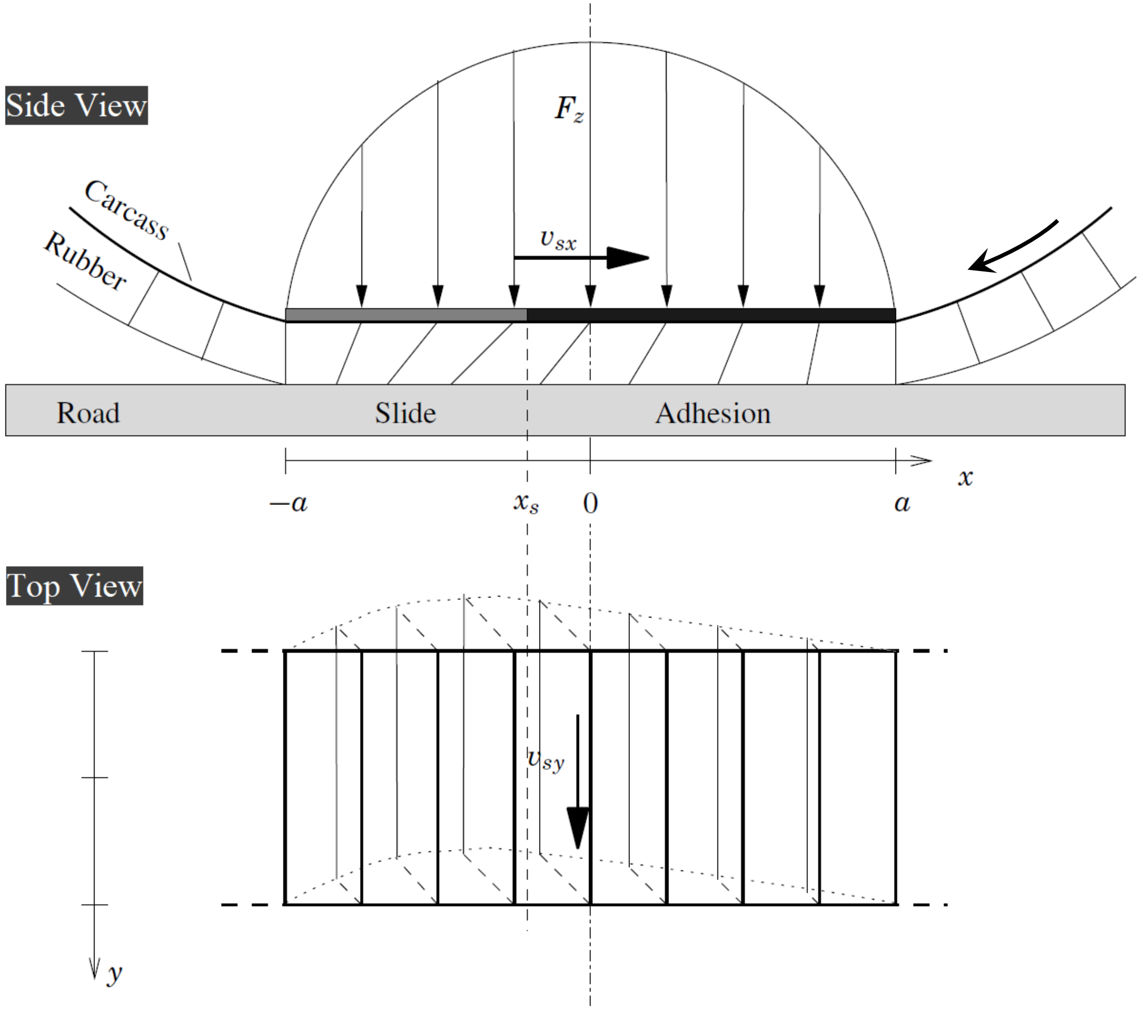
$$\delta_i = x_{ci} - x_{ri} = \int_0^t (V_x - \Omega r_e)\,\mathrm{d}t = \int_0^t V_{sx}\mathrm{d}t$$

Due to the short period brushes are in contact with ground, we can assume constant velocity

$$\delta_i = \frac{a - x_{ci}}{\Omega r_e} V_{sx} = (a - x_{ci}) \frac{V_x - \Omega r_e}{\Omega r_e} = (a - x_{ci})\,\sigma_x$$

Let's observe that there is a relation between theoretical and practical longitudinal slip

$$\sigma_x = \frac{\kappa}{1-\kappa}$$



Finally, assuming rubber has linear stiffness the force required to deform a brush is

$$F_{xi} = k\delta_i$$

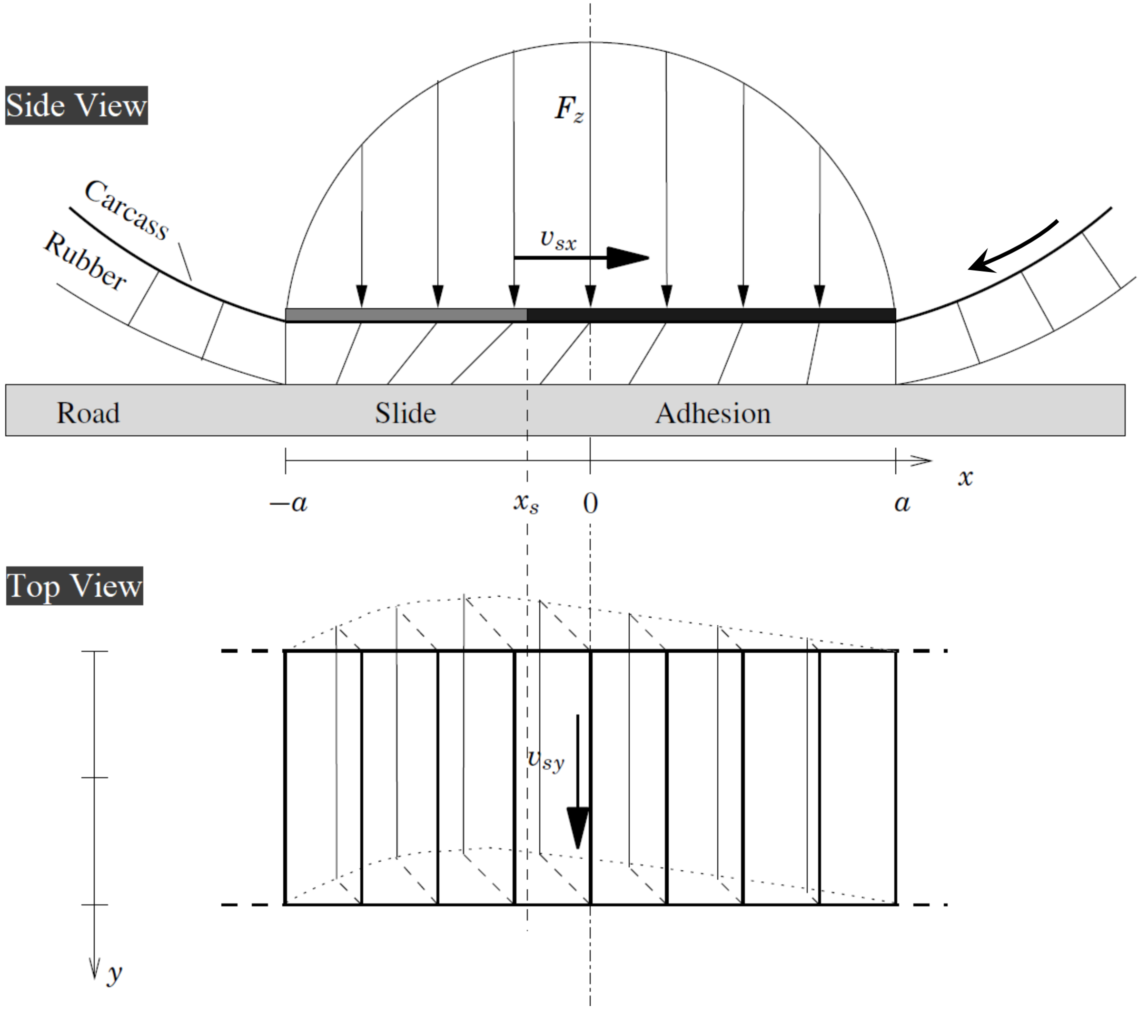
but this force is limited by the maximum force available at the brush tip due to static friction

$$F_{xi_{max}} = \mu F_{z_i}$$

at which corresponds a maximum deformation

$$\delta_{i_{max}} = \frac{\mu F_{z_i}}{k}$$

After reaching the maximum deformation brush start sliding.



Tire modelling: brush model derivation

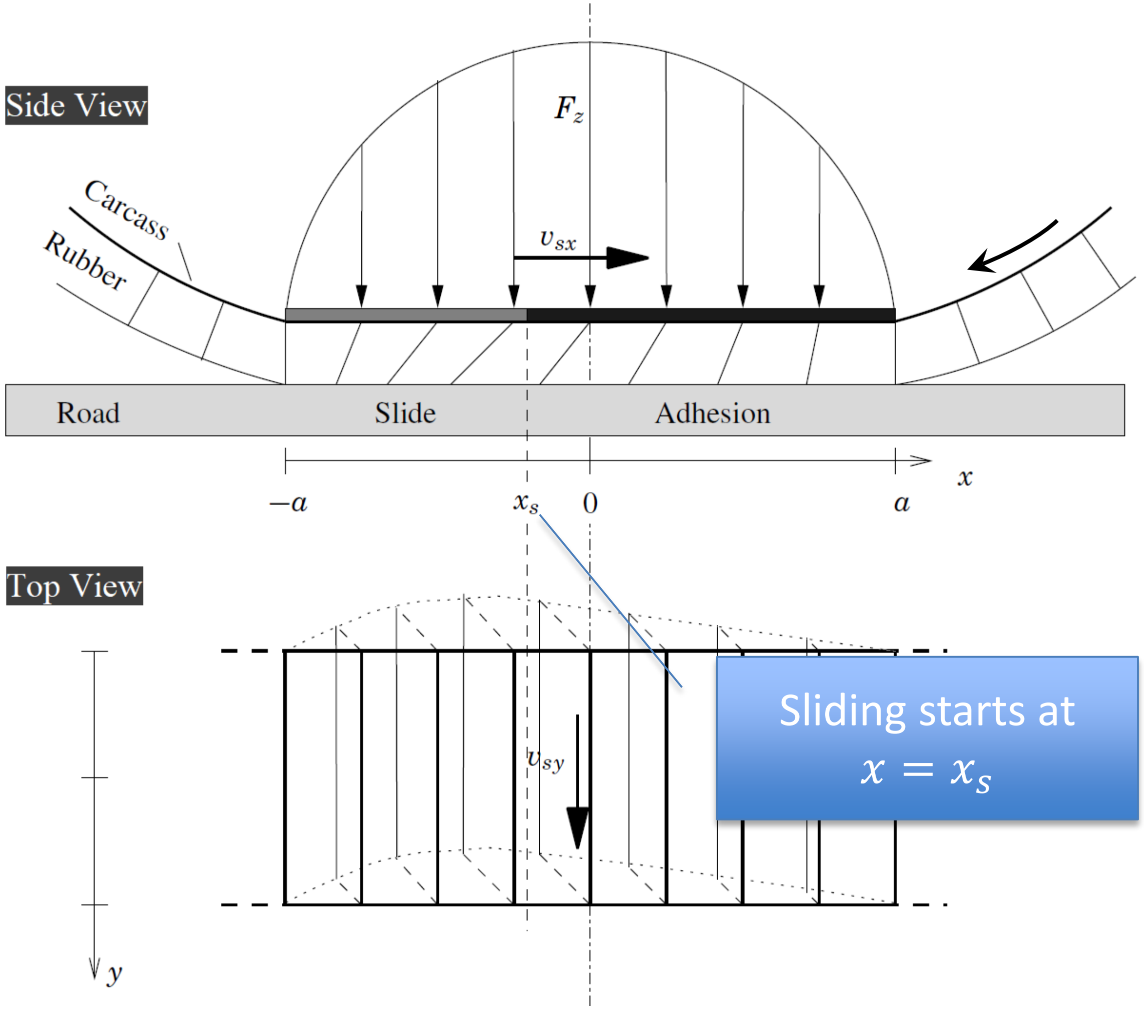
We assume that static and kinetic (sliding) friction coefficients are equal, the force acting on brushes during sliding is thus μF_{zi}.

We can have three different situations:

- adhesion in the entire contact patch
- both sliding and adhesion
- entire tire surface slides against ground

If we have both sliding and adhesion, the sliding starts at ($\delta_i = \delta_{i_{max}}$)

$$\sigma_x (a - x_{ci_s}) = \frac{\mu F_{z_i}}{k} \quad \Rightarrow \quad x_{ci_s} = a - \frac{\mu F_{z_i}}{\sigma_x k}$$



We now integrate over the all contact patch to determine the overall force.

We introduce a parabolic distribution for the normal force per unit of length

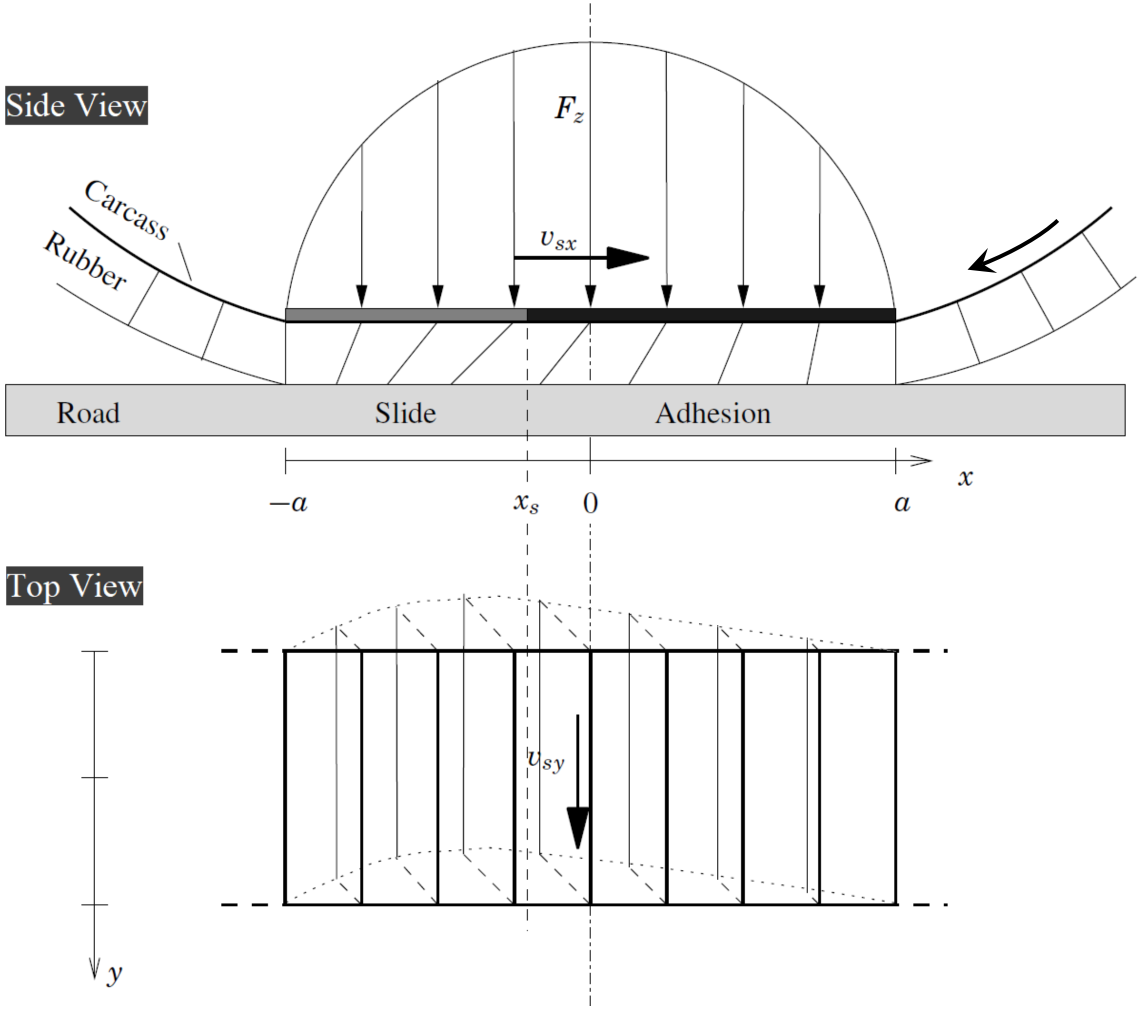
$$q_z(x_c) = q_M \left(1 - \frac{x_c^2}{a^2}\right)$$

and the normal force is given by

$$F_z = q_z(x_c)\,\mathrm{d}x_c$$

Integrating the pressure distribution along the patch

$$F_z = \int_{-a}^{+a} q_z(x_c)\,\mathrm{d}x_c = q_M \int_{-a}^{+a} \left(1 - \frac{x_c^2}{a^2}\right) \mathrm{d}x_c = \frac{4}{3} a q_M$$



Tire modelling: brush model derivation

From the previous integral we obtain

$$q_M = \frac{3F_z}{4a}$$

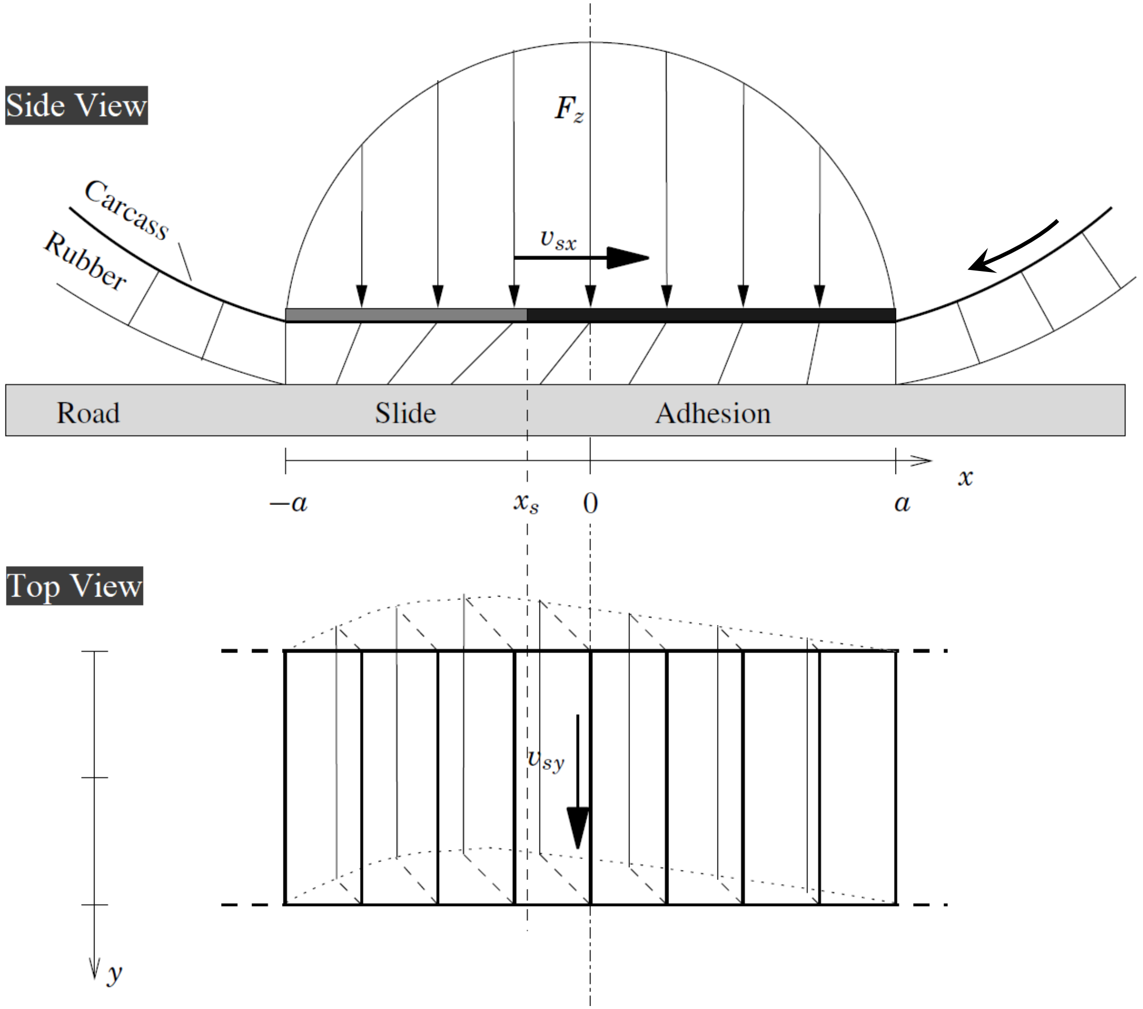
The pressure distribution is thus

$$q_z(x_c) = \frac{3F_z}{4a}\left(1 - \frac{x_c^2}{a^2}\right)$$

Adding the force during adhesion to the force during sliding we obtain the total longitudinal force

$$F_x = -\int_{-a}^{x_s} \mu q_x(x_c)\,\mathrm{d}x_c - \int_{x_s}^{a} c_p \sigma_x (a - x_c)\,\mathrm{d}x_c$$

where c_p sliding force dir adhesion force unit length ($k = c_p dx_c$).



Tire modelling: brush model derivation

Solving the integral we obtain

$$F_x = -\left(\frac{3\mu F_z}{4a}x_s + \frac{\mu F_z}{2} - \frac{\mu F_z}{4a^3}x_s^3\right)$$
$$-\left(c_p\sigma_x\frac{a^2}{2} + c_p\sigma_x\frac{x_s^2}{2} - c_p\sigma_x a x_s\right)$$

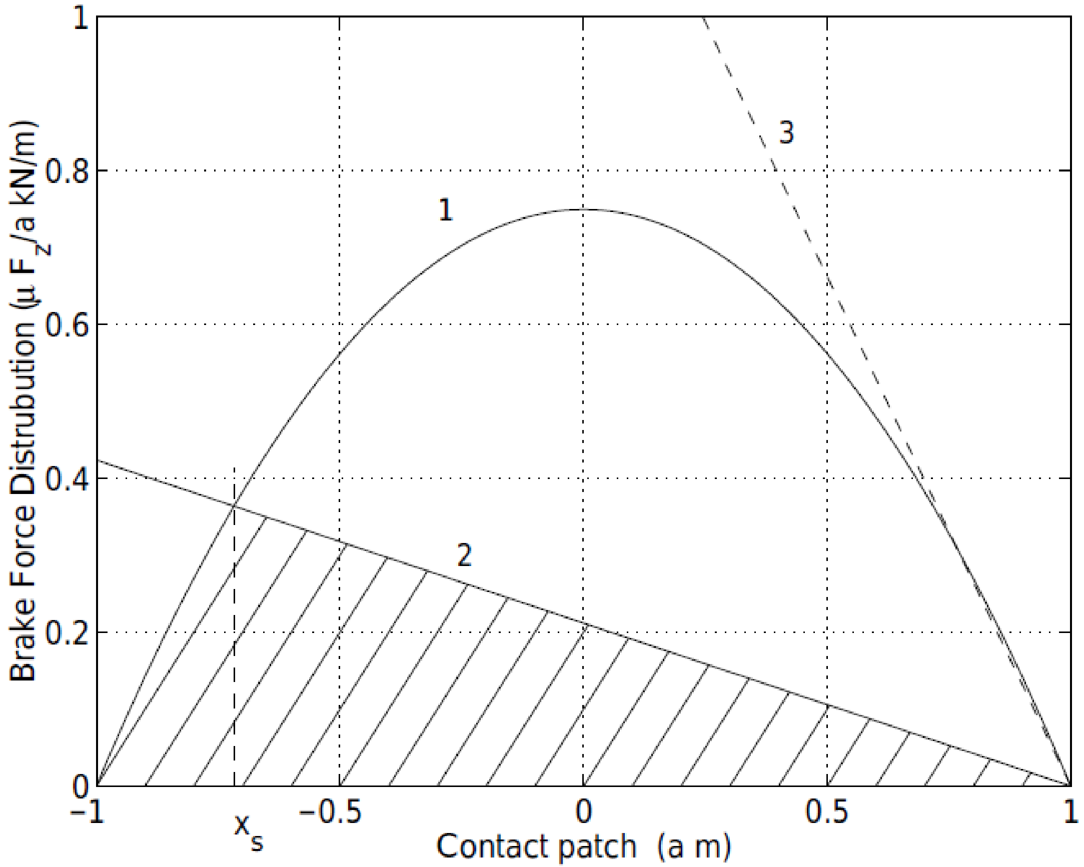
We compute the breakaway point x_s from

$$c_p\sigma_x\left(a - x_s\right) = \mu q_z\left(x_s\right)$$

That gives rise to

$$x_s^2 - \frac{4}{3}\frac{c_p\sigma_x a^3}{\mu F_z}x_s + \left(\frac{4}{3}\frac{c_p\sigma_x a^4}{\mu F_z} - a^2\right) = 0$$

The first solution $x_s = a$ is not acceptable, it means the tire is fully sliding.



Tire modelling: brush model derivation

The only acceptable solution is

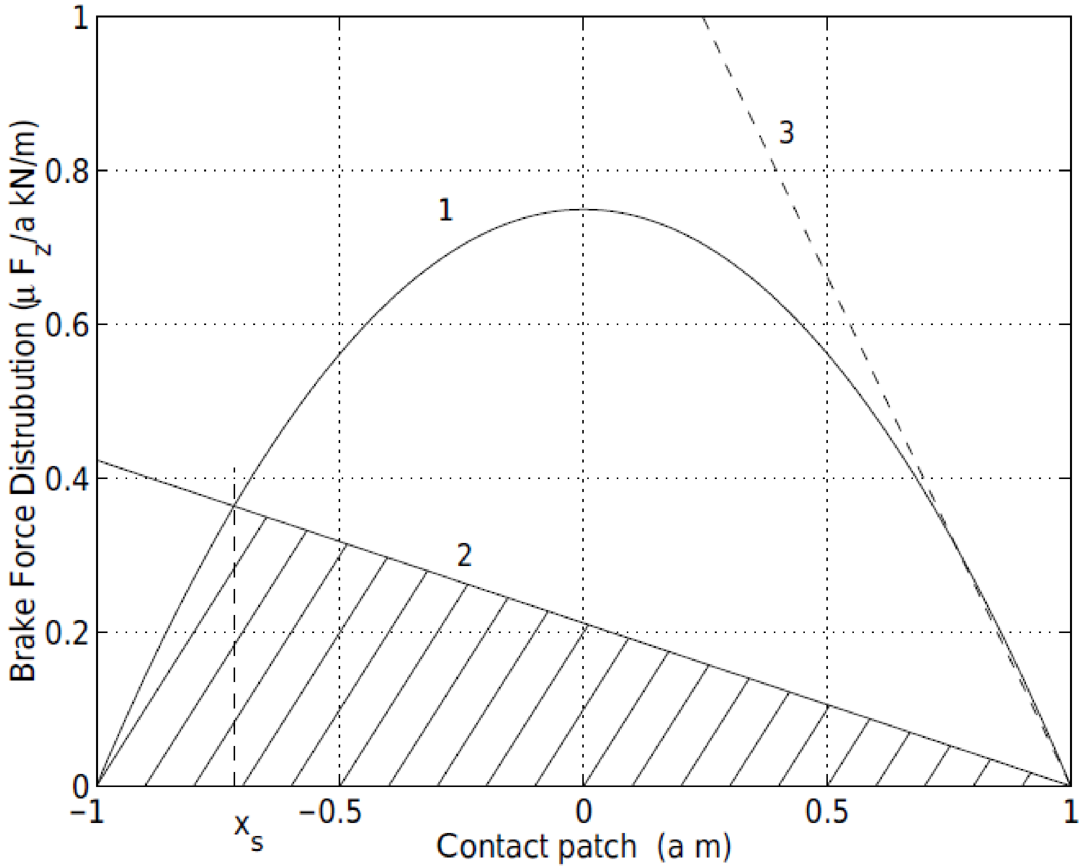
$$x_s = \frac{4}{3}\frac{c_p \sigma_x a^3}{\mu F_z} - a$$

Finally, introducing this relation into the solution of the integral we obtain

$$F_x = -2c_p a^2 \sigma_x + \frac{4}{3}\frac{\left(c_p a^2 \sigma_x\right)^2}{\mu F_z} - \frac{8}{27}\frac{\left(c_p a^2 \sigma_x\right)^3}{(\mu F_z)^2}$$

Some remarks:

- at low slip values the force-slip relation is almost linear
- if the tire is fully sliding $F_x = \mu F_z$



Tire modelling: brush model derivation

Recall the relation we use to compute x_s

$$c_p \sigma_x (a - x_s) = \mu q_z (x_s)$$

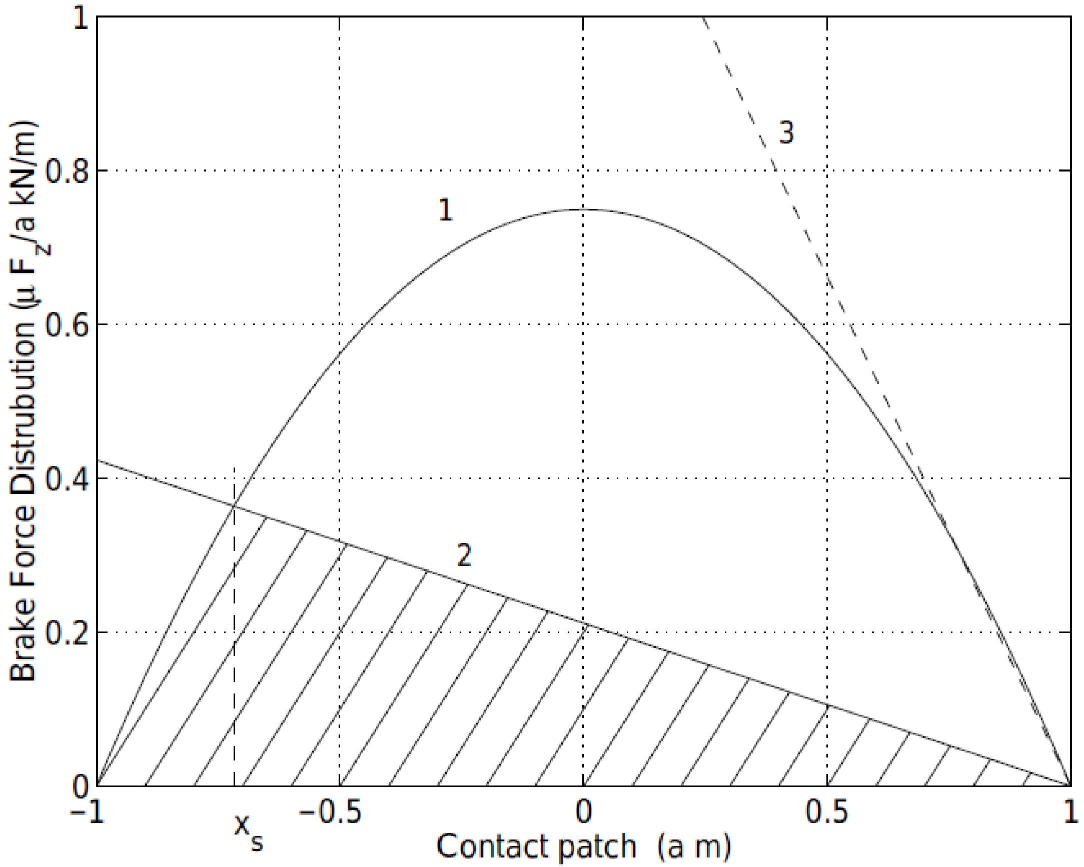
If slip increases the magnitude of the slope $c_p \sigma_x$ increases.

When tire starts sliding the line is tangent to the parabola in $x = a$, and the magnitude of the slope is

$$c_p \sigma_x = \frac{\mathrm{d}}{\mathrm{d}x_c} \left(\mu q_z (x_c) \right)_{x_c = a} = \frac{3}{2} \frac{\mu F_z}{a^2}$$

We thus conclude that the minimum slip value that gives full sliding is given by

$$\sigma_{x_{sl}} = \frac{3}{2} \frac{\mu F_z}{c_p a^2}$$



Tire modelling: brush model derivation

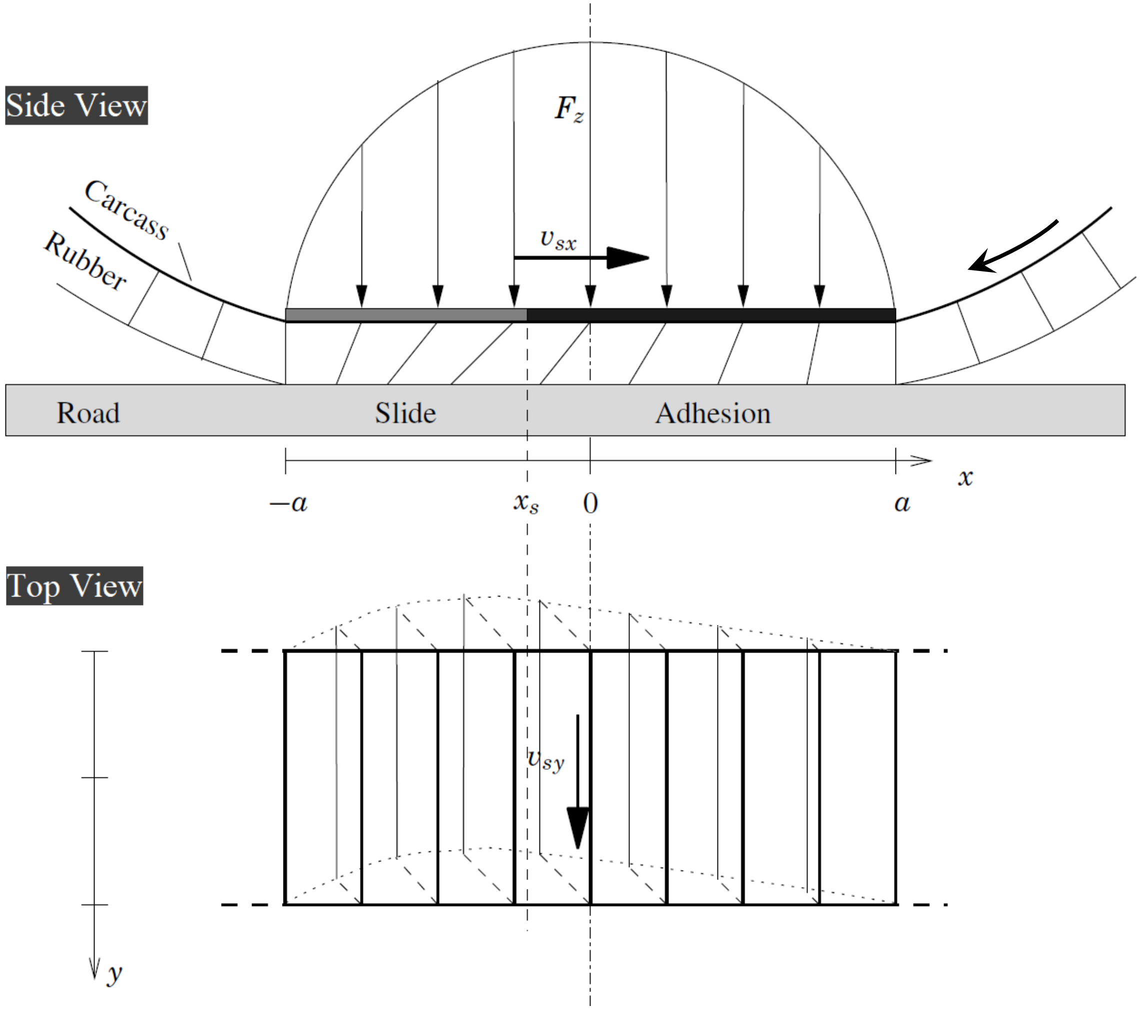
In conclusion, the longitudinal force is given by

$$F_x = \begin{cases} C_x \sigma_x \left(-1 + \frac{|\sigma_x|}{\sigma_{x_{sl}}} - \frac{\sigma_x^2}{3\sigma_{x_{sl}}^2} \right) & |\sigma_x| < \sigma_{x_{sl}} \\ -\mu F_z \operatorname{sign}(\sigma_x) & |\sigma_x| \geq \sigma_{x_{sl}} \end{cases}$$

where $C_x = 2c_p a^2$ is the *longitudinal stiffness* of the tire.

The slip σ_x is

- positive, when the vehicle is braking
- negative, when the vehicle is accelerating



Using the same procedure we can compute the lateral force

$$F_y = \begin{cases} C_\alpha z \left(-1 + \frac{|z|}{z_{sl}} - \frac{z^2}{3z_{sl}^2} \right) & |z| < z_{sl} \\ -\mu F_z \text{sign}(\alpha) & |z| \geq z_{sl} \end{cases}$$

where $C_\alpha = 2c_{p_y}a^2$ is the *cornering stiffness* of the tire and $z = \tan\alpha$.

This model is also called *Fiala tire model*.

Deriving the simplified dynamic model of a bicycle

We start again from the bicycle model we have already introduced

$$Ma_x = F_{x_F}\cos(\delta) - F_{y_F}\sin(\delta) + F_{x_R}$$

$$Ma_y = F_{x_F}\sin(\delta) + F_{y_F}\cos(\delta) + F_{y_R}$$

$$I_z\dot{r} = a\left(F_{x_F}\sin(\delta) + F_{y_F}\cos(\delta)\right) - bF_{y_R}$$

where

$$a_x = \dot{V}_x - rV_y = \dot{V}\cos(\beta) - V\left(\dot{\beta} + r\right)\sin(\beta)$$

$$a_y = \dot{V}_y - rV_x = \dot{V}\sin(\beta) + V\left(\dot{\beta} + r\right)\cos(\beta)$$

and if we assume small values for β ($\cos\beta \approx 1$, $\sin\beta \approx \beta$)

$$a_x = \dot{V} - V\left(\dot{\beta} + r\right)\beta \approx \dot{V}$$

$$a_y = \dot{V}\beta + V\left(\dot{\beta} + r\right) \approx V\left(\dot{\beta} + r\right)$$