Control of Mobile Robots

Prof. Bascetta

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SOLUTION

Control of Mobile Robots
Prof. Luca Bascetta

EXERCISE 1

1. Given the kinematic constraint

$$\dot{q}_1 - q_1\dot{q}_2 + 4\dot{q}_3 = 0$$

where $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ is the configuration vector. Determine, using the necessary and sufficient condition, if this constraint is holonomic or nonholonomic.

The necessary and sufficient condition for this constraint to be holonomic is that the first partial derivatives of $Q_1(\mathbf{q})$, $Q_2(\mathbf{q})$, $Q_3(\mathbf{q})$, with respect to q_1, q_2, and q_3 exist, and

$$\begin{aligned} \frac{\partial \alpha(\mathbf{q})}{\partial q_2} &= -\frac{\partial(\alpha(\mathbf{q})\, q_1)}{\partial q_1} = -\alpha(\mathbf{q}) - q_1 \frac{\partial \alpha(\mathbf{q})}{\partial q_1} \\ \frac{\partial \alpha(\mathbf{q})}{\partial q_3} &= 4\frac{\partial \alpha(\mathbf{q})}{\partial q_1} \\ 4\frac{\partial \alpha(\mathbf{q})}{\partial q_2} &= -\frac{\partial(\alpha(\mathbf{q})\, q_1)}{\partial q_3} = -q_1 \frac{\partial \alpha(\mathbf{q})}{\partial q_3} \end{aligned}$$

solving the previous relations we obtain

$$\begin{aligned} -0.25 q_1 \frac{\partial \alpha(\mathbf{q})}{\partial q_3} &= -\alpha(\mathbf{q}) - 0.25 q_1 \frac{\partial \alpha(\mathbf{q})}{\partial q_3} \\ \frac{\partial \alpha(\mathbf{q})}{\partial q_1} &= 0.25 \frac{\partial \alpha(\mathbf{q})}{\partial q_3} \\ \frac{\partial \alpha(\mathbf{q})}{\partial q_2} &= -0.25 q_1 \frac{\partial \alpha(\mathbf{q})}{\partial q_3} \end{aligned}$$

From the first relation we conclude $\alpha(\mathbf{q}) = 0$, and thus the constraint is nonholonomic.

2. Given the kinematic constraint

$$2\dot{q}_2 - q_1\dot{q}_3 = 0$$

where $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ is the configuration vector. Determine, using the necessary and sufficient condition, if this constraint is holonomic or nonholonomic.

The necessary and sufficient condition for this constraint to be holonomic is that the first partial derivatives of $Q_1(\mathbf{q})$, $Q_2(\mathbf{q})$, $Q_3(\mathbf{q})$, with respect to q_1, q_2, and q_3 exist, and

$$\begin{aligned} 0 &= 2\frac{\partial \alpha(\mathbf{q})}{\partial q_1} \\ 0 &= -\frac{\partial(\alpha(\mathbf{q})\, q_1)}{\partial q_1} = -\alpha(\mathbf{q}) - q_1 \frac{\partial \alpha(\mathbf{q})}{\partial q_1} \\ -\frac{\partial(\alpha(\mathbf{q})\, q_1)}{\partial q_2} &= -q_1 \frac{\partial \alpha(\mathbf{q})}{\partial q_2} = 2\frac{\partial \alpha(\mathbf{q})}{\partial q_3} \end{aligned}$$

From the first two relations we conclude $\alpha(\mathbf{q}) = 0$, and thus the constraint is nonholonomic.

3. Is the system of two constraints

$$\dot{q}_1 - q_1\dot{q}_2 + 4\dot{q}_3 = 0 \qquad 2\dot{q}_2 - q_1\dot{q}_3 = 0$$

holonomic or nonholonomic? Motivate the answer analysing the accessibility distribution.

We can rewrite the system of two constraints in Pfaffian form as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} 1 & -q_1 & 4 \\ 0 & 2 & -q_1 \end{bmatrix} \dot{\mathbf{q}} = 0$$

From the first two column it is straightforward to verify that rank $\left(A^T(\mathbf{q})\right) = 2$. As a consequence, a basis of the null space of $A^T(\mathbf{q})$ is composed by a single vector $g_1(\mathbf{q})$, and no other vector fields can be added to the accessibility distribution.
We thus conclude that the accessibility space has dimension 1, that is equal to $n-k$, and the system of constraints is holonomic.

EXERCISE 2
Consider a unicycle robot whose mass and yaw inertia vary with time.

1. Write the expression of the Lagrangian function, of matrix $S(\mathbf{q})$, and $A(\mathbf{q})$.

The unicycle configuration is represented by $\mathbf{q} = [x,\ y,\ \theta]$.
The kinetic energy is given by

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}M(t)\,\dot{x}^2 + \frac{1}{2}M(t)\,\dot{y}^2 + \frac{1}{2}I(t)\,\dot{\theta}^2$$

The potential energy, instead, has no contribution.
The Lagrangian function is thus

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}M(t)\,\dot{x}^2 + \frac{1}{2}M(t)\,\dot{y}^2 + \frac{1}{2}I(t)\,\dot{\theta}^2$$

The Pfaffian matrix for the pure rolling constraint is

$$A^T(\mathbf{q}) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix}$$

The inputs to the model are a force F and a torque τ. The projections of this force on x an y, and the torque τ make work on $\mathbf{q}$. Consequently, matrix $S(\mathbf{q})$ can be defined as

$$S(\mathbf{q}) = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}$$

2. Write the dynamic model using Lagrange equations.

   In order to compute the dynamic model we have to compute

$$G^T(\mathbf{q})\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}}\right)^T\right\} = G^T(\mathbf{q})\,S(\mathbf{q})\begin{bmatrix} F \\ \tau \end{bmatrix}$$

   where

$$G(\mathbf{q}) = \begin{bmatrix} \sin\theta & 0 \\ \cos\theta & 0 \\ 0 & 1 \end{bmatrix}$$

   First of all

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T = \frac{d}{dt}\left(\begin{bmatrix} M(t)\dot{x} \\ M(t)\dot{y} \\ I(t)\dot{\theta} \end{bmatrix}\right) = \begin{bmatrix} \dot{M}(t)\dot{x} + M(t)\ddot{x} \\ \dot{M}(t)\dot{y} + M(t)\ddot{y} \\ \dot{I}(t)\dot{\theta} + I(t)\ddot{\theta} \end{bmatrix}$$

   and

$$\left(\frac{\partial L}{\partial \mathbf{q}}\right)^T = \mathbf{0} \tag{1}$$

   Putting now everything together we obtain the dynamic model

$$\begin{aligned} \left(\dot{M}(t)\dot{x} + M(t)\ddot{x}\right)\sin\theta + \left(\dot{M}(t)\dot{y} + M(t)\ddot{y}\right)\cos\theta &= F \\ \dot{I}(t)\dot{\theta} + I(t)\ddot{\theta} &= \tau \end{aligned}$$

3. How does the time-varying mass and inertia affect the computation of the tyre-ground interaction model? Consider a linear interaction model.

   Assuming a linear interaction model, i.e., $F_y = C_\alpha \alpha$, one have to consider that the cornering stiffness C_α is no more constant, it is a function of the mass and thus it is time-varying.

EXERCISE 3

1. Write and explain the pseudocode of the algorithm to construct the probabilistic roadmap used by PRM.

   The PRM algorithm to construct the roadmap follows:

$V \leftarrow \emptyset$;
$E \leftarrow \emptyset$;
for $i = 1, \dots, N$ **do**
  $\mathbf{q}_{rand} \leftarrow SampleFree_i$;
  $U \leftarrow Near\left(G, \mathbf{q}_{rand}, r\right)$;
  $V \leftarrow V \cup \{\mathbf{q}_{rand}\}$;
  foreach $\mathbf{u} \in U$ *in order of increasing* $\|\mathbf{u} - \mathbf{q}_{rand}\|$ **do**
    if $\mathbf{q}_{rand}$ *and* $\mathbf{u}$ *are not in the same connected component of* G **then**
      if $CollisionFree\left(\mathbf{q}_{rand}, \mathbf{u}\right)$ **then**
        $E \leftarrow E \cup \{(\mathbf{q}_{rand}, \mathbf{u})\}$;
      end
    end
  end
end
return $G = (V, E)$

N vertex are sampled from the free space, then for each vertex $\mathbf{q}_{rand}$ the set of near nodes, i.e., the set of nodes in a ball of radius r centred in $\mathbf{q}_{rand}$, is computed and all the collision free connections between $\mathbf{q}_{rand}$ and the nodes in the near node set that are not in the same connected component are generated.

2. Explain how the previous algorithm has to be modified in order to obtain sPRM and PRM* algorithms.

The only difference between sPRM and PRM is that sPRM connects all the nodes in the near node set, without checking if they are in the same connected component.
The sPRM algorithm to construct the roadmap follows:

$V \leftarrow \{\mathbf{q}_{init}\} \cup \{SampleFree_i, i = 1, \dots, N\}$;
$E \leftarrow \emptyset$;
foreach $\mathbf{v} \in V$ **do**
  $U \leftarrow Near\left(G, \mathbf{v}, r\right) \setminus \{\mathbf{v}\}$;
  foreach $\mathbf{u} \in U$ **do**
    if $CollisionFree\left(\mathbf{v}, \mathbf{u}\right)$ **then**
      $E \leftarrow E \cup \{(\mathbf{v}, \mathbf{u})\}$;
    end
  end
end
return $G = (V, E)$

The only difference between PRM and PRM* is in the way the near set is computed. In PRM* the radius of the near neighbourhood is related to the number of sampled nodes, i.e.,

$$U \leftarrow Near\left(G, \mathbf{q}_{rand}, \gamma_{PRM}\left(\log\left(N\right)/N\right)^{1/d}\right) \setminus \{\mathbf{q}_{rand}\}$$

where d is the dimension of the configuration space, and γ_{PRM} is a suitable constant.

3. In PRM the Near function is used to determine the nodes that belong to the Nearest neighbour. Give a mathematical definition of the Near function used by PRM, explaining how the function works. Show two other ways of computing the Nearest neighbour.

The Near function used by PRM is defined in this way

$$\text{Near} : (G, \mathbf{q}, r) \rightarrow V' \subset V$$

Given a set of vertices V and a vertex $\mathbf{q} \in V$, the Near function returns all the vertices in V that are contained in a ball of radius r centred at $\mathbf{q}$.
There are two other ways to determine the Nearest neighbour:

$$\text{Nearest} : (G, \mathbf{q}) = \arg\min_{\mathbf{v} \in V} \|\mathbf{q} - \mathbf{v}\|$$

and

$$\text{kNearest} : (G, \mathbf{q}, k) = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$$

that returns the k vertices in V that are the nearest to $\mathbf{q}$.

EXERCISE 4

Consider a simplified version of the rear-wheel drive bicycle model

$$\begin{aligned} \dot{x} &= v \cos\theta \\ \dot{y} &= v \sin\theta \\ \dot{\theta} &= \frac{v}{\ell} \tan\phi \end{aligned}$$

where (x, y, θ) is the position and orientation of the vehicle, v the linear velocity, and ϕ the steering angle.

1. Write the expression of the feedback linearising law for this model.

When can use the same linearising law given for the unicycle

$$\begin{aligned} v &= v_{xP} \cos\theta + v_{yP} \sin\theta \\ \omega &= \frac{1}{\epsilon} \left(v_{yP} \cos\theta - v_{xP} \sin\theta \right) \end{aligned}$$

including the change of variables $\omega = \frac{v}{\ell} \tan\phi$. We obtain

$$\begin{aligned} v &= v_{xP} \cos\theta + v_{yP} \sin\theta \\ \phi &= \arctan\left(\frac{\ell}{\epsilon} \frac{v_{yP} \cos\theta - v_{xP} \sin\theta}{v_{xP} \cos\theta + v_{yP} \sin\theta} \right) \end{aligned}$$

2. Is the previous linearising feedback affected by any singularity? Does it introduce any hidden dynamics? If yes, which are the states that belong to the hidden dynamics? Clearly motivate the answer.

The linearising feedback is singular for $v = 0$.
The linearising feedback induces an hidden dynamics that is composed by the heading state θ.

3. Write the equations of the dynamical system representing the closed-loop system obtained connecting the model with the controller.

The closed-loop system is described by the following dynamical system

$$\begin{aligned}
\dot{x} &= v_{xP} \cos^2\theta + v_{yP} \sin\theta\cos\theta \\
\dot{y} &= v_{xP} \cos\theta\sin\theta + v_{yP} \sin^2\theta \\
\dot{\theta} &= \frac{1}{\epsilon}\left(v_{yP} \cos\theta - v_{xP} \sin\theta\right)
\end{aligned}$$