Control of Industrial and Mobile Robots

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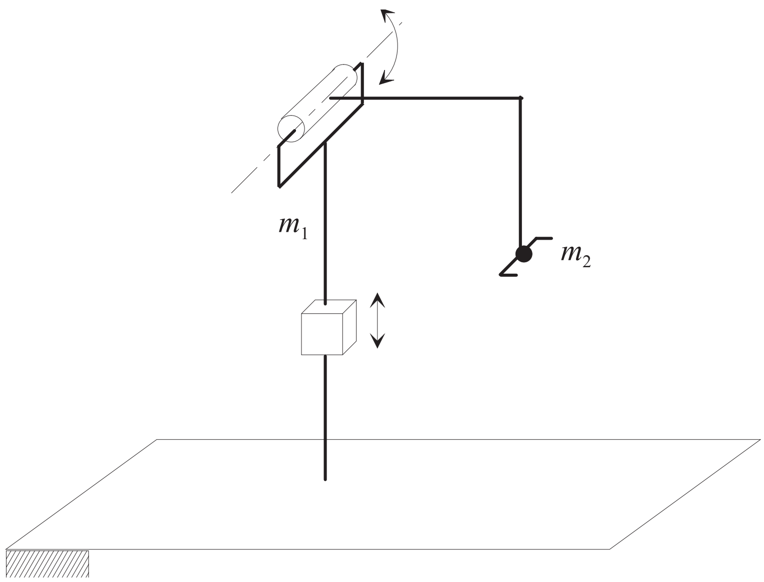
SOLUTION

EXERCISE 1

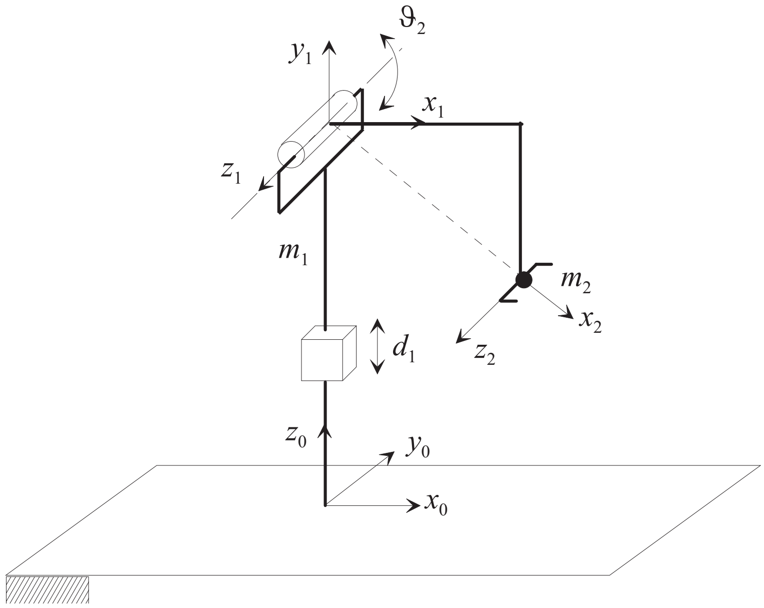
1. Consider the manipulator sketched in the picture, where the mass of the second link is assumed to be concentrated at the end-effector:



Find the expression of the inertia matrix $\mathbf{B}(\mathbf{q})$ of the manipulator[1].

Denavit-Hartenberg frames can be defined as sketched in this picture:

[1]The cross product between vector $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is $c = a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$



Computations of the Jacobians:

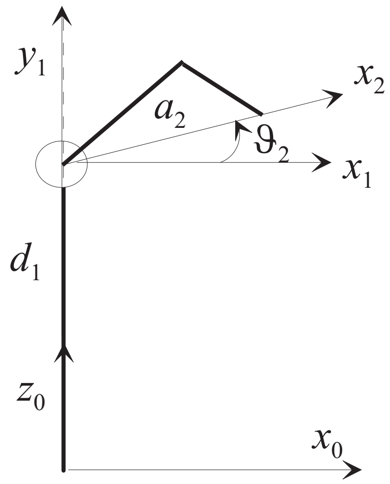
Link 1

$$\mathbf{J}_P^{(l_1)} = \left[\begin{array}{cc} \mathbf{j}_{P_1}^{(l_1)} & \mathbf{0} \end{array} \right] = \left[\begin{array}{cc} \mathbf{z}_0 & \mathbf{0} \end{array} \right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Link 2

$$\mathbf{J}_P^{(l_2)} = \left[\begin{array}{cc} \mathbf{j}_{P_1}^{(l_2)} & \mathbf{j}_{P_2}^{(l_2)} \end{array} \right] = \left[\begin{array}{cc} \mathbf{z}_0 & \mathbf{z}_1 \times (\mathbf{p}_{l_2} - \mathbf{p}_1) \end{array} \right] = \begin{bmatrix} 0 & -a_2 s_2 \\ 0 & 0 \\ 1 & a_2 c_2 \end{bmatrix}$$

For the above computations, we can make reference to the following picture:



and to the following auxiliary vectors:

$$\mathbf{p}_{l_2} = \begin{bmatrix} a_2 c_2 \\ 0 \\ d_1 + a_2 s_2 \end{bmatrix}, \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}, \mathbf{z}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

The inertia matrix can be computed now:

$$\begin{aligned} \mathbf{B}(\mathbf{q}) &= m_1 \mathbf{J}_P^{(l_1)^T} \mathbf{J}_P^{(l_1)} + m_2 \mathbf{J}_P^{(l_2)^T} \mathbf{J}_P^{(l_2)} + \\ &= m_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} 1 & a_2 c_2 \\ a_2 c_2 & a_2^2 \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \end{aligned}$$

where:

$$\begin{aligned} b_{11} &= m_1 + m_2 \\ b_{12} &= m_2 a_2 c_2 \\ b_{22} &= m_2 a_2^2 \end{aligned}$$

2. Compute the matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ of the Coriolis and centrifugal terms[2] for this manipulator.

The only derivative in the Christoffel symbols which is different from zero is:

$$\frac{\partial b_{12}}{\partial q_2} = \frac{\partial b_{21}}{\partial q_2} = -m_2 a_2 s_2$$

therefore

$$c_{111} = 0 \qquad c_{211} = 0$$

$$c_{112} = c_{121} = 0 \qquad c_{212} = c_{221} = \frac{1}{2}\left(\frac{\partial b_{21}}{\partial q_2} - \frac{\partial b_{12}}{\partial q_2}\right) = 0$$

$$c_{122} = \frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_2} + \frac{\partial b_{21}}{\partial q_2}\right) = -m_2 a_2 s_2 \qquad c_{222} = 0$$

The matrix of the Coriolis and centrifugal terms is thus:

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

where:

$$\begin{aligned} c_{11} &= c_{111}\dot{q}_1 + c_{112}\dot{q}_2 = 0 \\ c_{12} &= c_{121}\dot{q}_1 + c_{122}\dot{q}_2 = -m_2 a_2 s_2 \dot{\vartheta}_2 \\ c_{21} &= c_{211}\dot{q}_1 + c_{212}\dot{q}_2 = 0 \\ c_{22} &= c_{221}\dot{q}_1 + c_{222}\dot{q}_2 = 0 \end{aligned}$$

[2] The general expression of the Christoffel symbols is $c_{ijk} = \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i}\right)$

3. Ignoring the gravitational terms, write the dynamic model for this manipulator.

   The model is formed by the equations:

$$
\begin{aligned}
(m_1 + m_2)\,\ddot{d}_1 + m_2 a_2 c_2 \ddot{\vartheta}_2 - m_2 a_2 s_2 \dot{\vartheta}_2^2 &= \tau_1 \\
m_2 a_2 c_2 \ddot{d}_1 + m_2 a_2^2 \ddot{\vartheta}_2 &= \tau_2
\end{aligned}
$$

4. Show that the model obtained in the previous step is linear with respect to a set of dynamic parameters.

   The model can be written in the form:

$$\mathbf{Y}\left(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\right)\mathbf{\Pi} = \tau$$

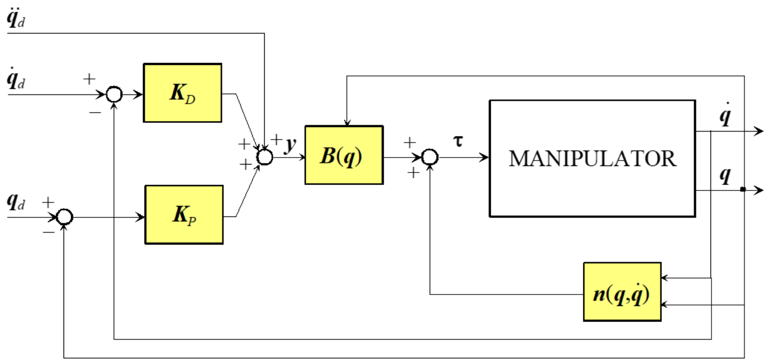
   with:

$$\mathbf{\Pi} = \begin{bmatrix} m_1 + m_2 \\ m_2 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} \ddot{d}_1 & a_2 c_2 \ddot{\vartheta}_2 - a_2 s_2 \dot{\vartheta}_2^2 \\ 0 & a_2 c_2 \ddot{d}_1 + a_2^2 \ddot{\vartheta}_2 \end{bmatrix}$$

EXERCISE 2

1. Consider the control scheme sketched in the following picture:



   Explain which control scheme it refers to and what is the result in terms of closed-loop dynamics that can be achieved with such control scheme.

   The control scheme is an inverse dynamics controller. The result that is achieved is the following closed loop dynamics:

$$\ddot{\tilde{\mathbf{q}}} + \mathbf{K}_D \dot{\tilde{\mathbf{q}}} + \mathbf{K}_P \tilde{\mathbf{q}} = 0$$

where $\tilde{\mathbf{q}} = \mathbf{q}_d - \mathbf{q}$ is the error in joint space.

2. In a two-link planar manipulator in the vertical plane with prismatic joints, the inertia matrix and the gravitational terms take the following expressions, respectively:

$$\begin{aligned} \mathbf{B} &= \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix} \\ \mathbf{g} &= \begin{bmatrix} (m_1 + m_2)\, g \\ 0 \end{bmatrix} \end{aligned}$$

Write the expression (equation by equation) of the control law for the control scheme of this exercise, specific for this manipulator.

In this case, the equations are written as:

$$\begin{aligned} \tau_1 &= (m_1 + m_2)\, y_1 + (m_1 + m_2)\, g \\ \tau_2 &= m_2 y_2 \end{aligned}$$

and:

$$\begin{aligned} y_1 &= \ddot{q}_{d1} + K_{D1}\left(\dot{q}_{d1} - \dot{q}_1\right) + K_{P1}\left(q_{d1} - q_1\right) \\ y_2 &= \ddot{q}_{d2} + K_{D2}\left(\dot{q}_{d2} - \dot{q}_2\right) + K_{P2}\left(q_{d2} - q_2\right) \end{aligned}$$

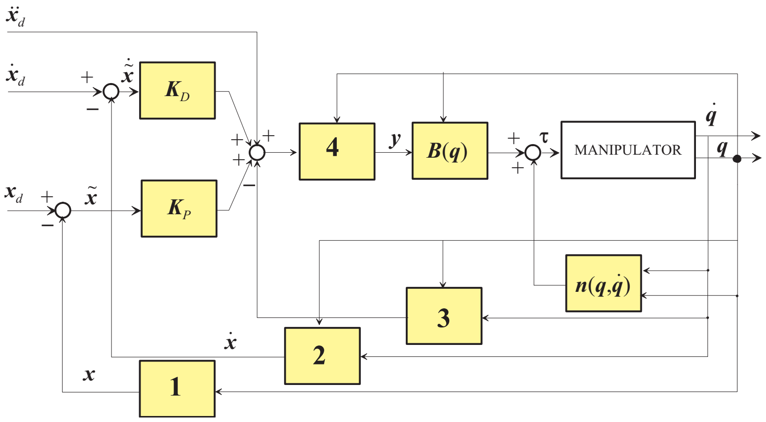
3. Tune the two matrices $\mathbf{K}_P$ and $\mathbf{K}_D$ in such a way that the dynamics of the error in the two joints is identical with two real eigenvalues at frequency 10 rad/s

We can select $\mathbf{K}_P = k_P \mathbf{I}_2$ and $\mathbf{K}_D = k_D \mathbf{I}_2$ such that:

$$s^2 + k_D s + K_P = (s + 10)^2 = s^2 + 20s + 100$$

Therefore $k_D = 20$ and $k_P = 100$.

4. In the operational space version of the control scheme of this exercise, four more blocks appear, numbered as 1, 2, 3, 4 in the following sketch:



Without adding any further comment, write the mathematical expressions of the blocks 1, 2, 3, 4.

1: $\mathbf{k}(\cdot)$ (direct kinematics)
2: $\mathbf{J}_A(\mathbf{q})$
3: $\dot{\mathbf{J}}_A(\mathbf{q}, \dot{\mathbf{q}})$
4: $\mathbf{J}_A^{-1}(\mathbf{q})$

EXERCISE 3

1. Given the kinematic constraint
$$\dot{q}_1 - q_1\dot{q}_2 + 4\dot{q}_3 = 0$$
where $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ is the configuration vector. Determine, using the necessary and sufficient condition, if this constraint is holonomic or nonholonomic.

The necessary and sufficient condition for this constraint to be holonomic is that the first partial derivatives of $Q_1(\mathbf{q})$, $Q_2(\mathbf{q})$, $Q_3(\mathbf{q})$, with respect to q_1, q_2, and q_3 exist, and
$$\frac{\partial \alpha(\mathbf{q})}{\partial q_2} = -\frac{\partial(\alpha(\mathbf{q})\,q_1)}{\partial q_1} = -\alpha(\mathbf{q}) - q_1\frac{\partial \alpha(\mathbf{q})}{\partial q_1}$$
$$\frac{\partial \alpha(\mathbf{q})}{\partial q_3} = 4\frac{\partial \alpha(\mathbf{q})}{\partial q_1}$$
$$4\frac{\partial \alpha(\mathbf{q})}{\partial q_2} = -\frac{\partial(\alpha(\mathbf{q})\,q_1)}{\partial q_3} = -q_1\frac{\partial \alpha(\mathbf{q})}{\partial q_3}$$
solving the previous relations we obtain
$$-0.25q_1\frac{\partial \alpha(\mathbf{q})}{\partial q_3} = -\alpha(\mathbf{q}) - 0.25q_1\frac{\partial \alpha(\mathbf{q})}{\partial q_3}$$
$$\frac{\partial \alpha(\mathbf{q})}{\partial q_1} = 0.25\frac{\partial \alpha(\mathbf{q})}{\partial q_3}$$
$$\frac{\partial \alpha(\mathbf{q})}{\partial q_2} = -0.25q_1\frac{\partial \alpha(\mathbf{q})}{\partial q_3}$$
From the first relation we conclude $\alpha(\mathbf{q}) = 0$, and thus the constraint is nonholonomic.

2. Given the kinematic constraint

$$2\dot{q}_2 - q_1\dot{q}_3 = 0$$

where $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ is the configuration vector. Determine, using the necessary and sufficient condition, if this constraint is holonomic or nonholonomic.

The necessary and sufficient condition for this constraint to be holonomic is that the first partial derivatives of $Q_1(\mathbf{q})$, $Q_2(\mathbf{q})$, $Q_3(\mathbf{q})$, with respect to q_1, q_2, and q_3 exist, and

$$\begin{aligned} 0 &= 2\frac{\partial \alpha(\mathbf{q})}{\partial q_1} \\ 0 &= -\frac{\partial(\alpha(\mathbf{q})\, q_1)}{\partial q_1} = -\alpha(\mathbf{q}) - q_1\frac{\partial \alpha(\mathbf{q})}{\partial q_1} \\ -\frac{\partial(\alpha(\mathbf{q})\, q_1)}{\partial q_2} &= -q_1\frac{\partial \alpha(\mathbf{q})}{\partial q_2} = 2\frac{\partial \alpha(\mathbf{q})}{\partial q_3} \end{aligned}$$

From the first two relations we conclude $\alpha(\mathbf{q}) = 0$, and thus the constraint is nonholonomic.

3. Is the system of two constraints

$$\dot{q}_1 - q_1\dot{q}_2 + 4\dot{q}_3 = 0 \qquad 2\dot{q}_2 - q_1\dot{q}_3 = 0$$

holonomic or nonholonomic? Motivate the answer analysing the accessibility distribution.

We can rewrite the system of two constraints in Pfaffian form as

$$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} 1 & -q_1 & 4 \\ 0 & 2 & -q_1 \end{bmatrix} \dot{\mathbf{q}} = 0$$

From the first two column it is straightforward to verify that $\mathrm{rank}\left(A^T(\mathbf{q})\right) = 2$. As a consequence, a basis of the null space of $A^T(\mathbf{q})$ is composed by a single vector $g_1(\mathbf{q})$, and no other vector fields can be added to the accessibility distribution.
We thus conclude that the accessibility space has dimension 1, that is equal to $n-k$, and the system of constraints is holonomic.

4. Considering the two constraints separately, are these

$$\begin{aligned} \dot{q}_1 &= q_1u_1 - 4u_2 \\ \dot{q}_2 &= u_1 \\ \dot{q}_3 &= u_2 \end{aligned} \qquad\qquad \begin{aligned} \dot{q}_1 &= u_2 \\ \dot{q}_2 &= 0.5q_1u_1 \\ \dot{q}_3 &= u_1 \end{aligned}$$

the kinematic models associated to the constraints in 1 and 2? Clearly motivate the answer.

The first constraint in Pfaffian form is represeted by

$$A^T(\mathbf{q}) = \begin{bmatrix} 1 & -q_1 & 4 \end{bmatrix}$$

The first kinematic model is represented by the following G matrix

$$G(\mathbf{q}) = \begin{bmatrix} q_1 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

whose rank is equal to two. The columns of this matrix are thus two independent vectors, and belong to the null space of $A^T(\mathbf{q})$, i.e., $A^T(\mathbf{q})\,\mathbf{g}_1 = 0$ and $A^T(\mathbf{q})\,\mathbf{g}_2 = 0$. We thus conclude that this is the kinematic model associated to the constraint represented by $A^T(\mathbf{q})$.
The second constraint in Pfaffian form is represeted by

$$A^T(\mathbf{q}) = \begin{bmatrix} 0 & 2 & -q_1 \end{bmatrix}$$

The second kinematic model is represented by the following G matrix

$$G(\mathbf{q}) = \begin{bmatrix} 0 & 1 \\ \frac{q_1}{2} & 0 \\ 1 & 0 \end{bmatrix}$$

whose rank is equal to two. The columns of this matrix are thus two independent vectors, and belong to the null space of $A^T(\mathbf{q})$, i.e., $A^T(\mathbf{q})\,\mathbf{g}_1 = 0$ and $A^T(\mathbf{q})\,\mathbf{g}_2 = 0$. We thus conclude that this is the kinematic model associated to the constraint represented by $A^T(\mathbf{q})$.

EXERCISE 4

Consider a simplified version of the rear-wheel drive bicycle model

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \frac{v}{\ell}\tan\phi \end{aligned}$$

where (x, y, θ) is the position and orientation of the vehicle, v the linear velocity, and ϕ the steering angle.

1. Write the expression of the feedback linearising law for this model.

When can use the same linearising law given for the unicycle

$$\begin{aligned} v &= v_{xP}\cos\theta + v_{yP}\sin\theta \\ \omega &= \frac{1}{\epsilon}\left(v_{yP}\cos\theta - v_{xP}\sin\theta\right) \end{aligned}$$

including the change of variables $\omega = \frac{v}{\ell}\tan\phi$. We obtain

$$\begin{aligned} v &= v_{xP}\cos\theta + v_{yP}\sin\theta \\ \phi &= \arctan\left(\frac{\ell}{\epsilon}\frac{v_{yP}\cos\theta - v_{xP}\sin\theta}{v_{xP}\cos\theta + v_{yP}\sin\theta}\right) \end{aligned}$$

2. Is the previous linearising feedback affected by any singularity? Does it introduce any hidden dynamics? If yes, which are the states that belong to the hidden dynamics? Clearly motivate the answer.

   The linearising feedback is singular for $v = 0$.
   The linearising feedback induces an hidden dynamics that is composed by the heading state θ.

3. Write the equations of the dynamical system representing the closed-loop system obtained connecting the model with the controller.

   The closed-loop system is described by the following dynamical system

$$
\begin{aligned}
\dot{x} &= v_{xP} \cos^2\theta + v_{yP} \sin\theta \cos\theta \\
\dot{y} &= v_{xP} \cos\theta \sin\theta + v_{yP} \sin^2\theta \\
\dot{\theta} &= \frac{1}{\epsilon} \left(v_{yP} \cos\theta - v_{xP} \sin\theta \right)
\end{aligned}
$$

4. Assuming as inputs

$$
v_{xP}(t) = \bar{v}_P \cos\bar{\theta}_P \qquad v_{yP}(t) = \bar{v}_P \sin\bar{\theta}_P
$$

   with $\bar{v}_P$ and $\bar{\theta}_P$ constants, and $\bar{v}_P > 0$.
   Study the stability of the hidden state.

   The dynamics of the hidden state is described by

$$
\dot{\theta} = \frac{1}{\epsilon} \left(\bar{v}_P \sin\bar{\theta}_P \cos\theta - \bar{v}_P \cos\bar{\theta}_P \sin\theta \right) = -\frac{\bar{v}_P}{\epsilon} \sin\left(\theta - \theta_P\right)
$$

   Defining $\Delta\theta = \theta - \theta_P$ we obtain

$$
\dot{\Delta\theta} = -\frac{\bar{v}_P}{\epsilon} \sin\left(\Delta\theta\right)
$$

   This system has two families of equilibria:

   - $\theta = \theta_P + 2k\pi$, $k \in \mathbb{Z}$, that are asymptotically stable;
   - $\theta = \theta_P + (2k+1)\pi$, $k \in \mathbb{Z}$, that are unstable.