Automatic Control
Exercise 3: Stability and performance analysis of closed-loop systems
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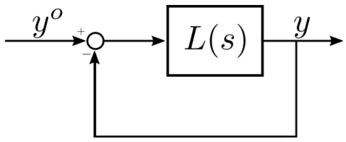
Exercise 1

Given the following system



where

$$L(s) = \frac{0.01}{(s+5)^2}$$

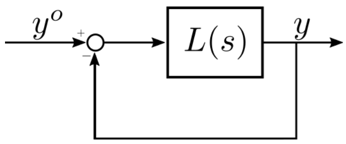
study the stability of the unitary negative feedback closed-loop system.

Solution

The magnitude of the loop transfer function is always less than 1 (less than $0\,dB$), the crossover frequency cannot be defined and, consequently, the Bode criterion cannot be applied.
Being the magnitude of the loop transfer function always less than 1 the Nyquist plot cannot make encirclements around -1. Applying the Nyquist criterion we conclude that $N = 0$ and $P_d = 0$, and the closed-loop system is thus asymptotically stable.

Exercise 2

Given the following system



where

$$L(s) = \frac{50}{s-5}$$

study the stability of the unitary negative feedback closed-loop system.

Solution

The loop transfer function has a pole in the right half plane, we cannot apply the Bode criterion.
Using magnitude and phase Bode plots (Fig. 1) we can draw the Nyquist plot (Fig. 2).

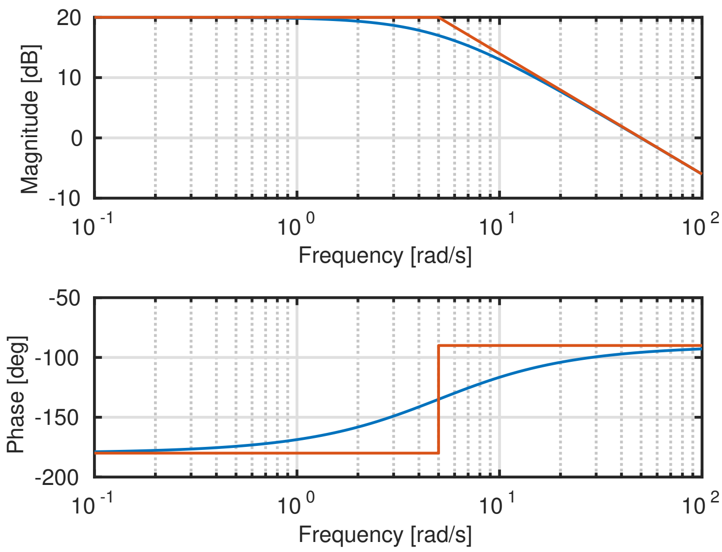


Figure 1: Loop transfer function Bode plot

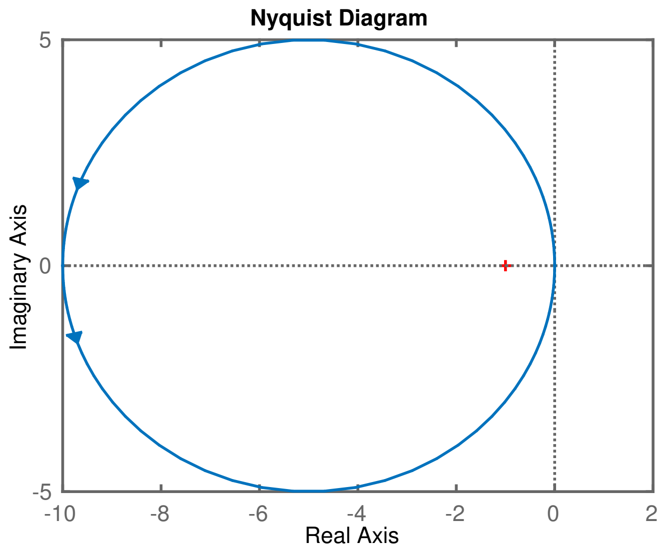
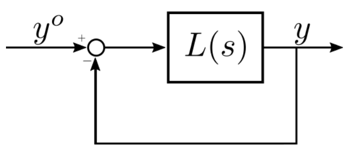


Figure 2: Loop transfer function Nyquist plot

From the Nyquist criterion it follows that the closed-loop system is asymptotically stable ($N = P_d = 1$).

Exercise 3

Given the following system



where

$$L(s) = \frac{10}{(1+0.1s)^2(1+0.01s)}e^{-\tau s}$$

study the stability of the unitary negative feedback closed-loop system for $\tau = 0$, and find the maximum delay τ for which the closed-loop system remains asymptotically stable.

Solution

From the magnitude Bode diagram of the loop transfer function (Fig. 3) it follows that the crossover frequency is approximately equal to $30\,rad/s$. The phase at the crossover frequency is given by

$$\varphi_c = -2\arctan(3) - \arctan(0.3) \approx -159.8^o$$

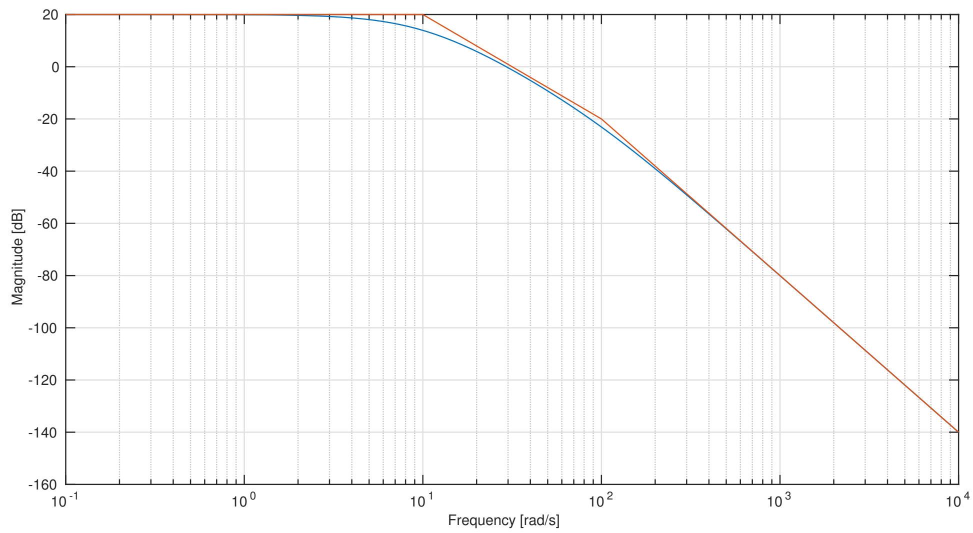


Figure 3: Loop transfer function Bode plot

As a consequence the phase margin is

$$\varphi_m = 180^o - \|\varphi_c\| \approx 20.2^o$$

and the system is asymptotically stable.
Considering now the delay, it causes a decrement of the phase margin given by

$$-30\tau\frac{180^o}{\pi}$$

The maximum delay τ for which the closed-loop system remains asymptotically stable is thus given by

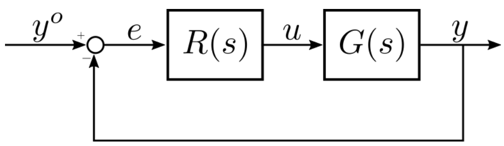
$$20.2^o - 30\tau\frac{180^o}{\pi} > 0$$

from which it follows

$$\tau < \frac{20.2^o}{30}\frac{\pi}{180^o} = 1.17\,s$$

Exercise 4

Given the following system



where

$$R(s) = \frac{1-s}{1+s} \qquad G(s) = \frac{1}{1-s}$$

analyse the stability of the closed-loop system.

Solution

The loop transfer function is

$$L(s) = R(s)G(s) = \frac{1-s}{1+s}\frac{1}{1-s} = \frac{1}{1+s}$$

Due to the pole-zero cancellation between two singularities in the right half plane, creating an unstable hidden dynamic, the closed-loop system cannot be asymptotically stable.
In order to understand what happens when we close the loop, we study the stability of the closed-loop system in the state space.
We start introducing a realization of $R(s)$ and $G(s)$.
Applying Laplace transform properties, from $G(s)$ we obtain

$$(1-s)Y(s) = U(s) \qquad \Rightarrow \qquad y(t) - \dot{y}(t) = u(t)$$

and in the state space

$$\begin{aligned} \dot{x}(t) &= x(t) - u(t) \\ y(t) &= x(t) \end{aligned}$$

The controller transfer function can be rewritten as

$$R(s) = \frac{2}{1+s} - 1$$

in order to decompose it into a strictly proper system and a constant. Following now the same procedure used for the plant transfer function we obtain the following state space model

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) \\ y(t) &= 2x(t) - u(t) \end{aligned}$$

Putting together the two realization we obtain the realization of the loop transfer function

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + e(t) \\ \dot{x}_2(t) &= 2x_1(t) + x_2(t) - e(t) \\ y(t) &= -x_2(t) \end{aligned}$$

Finally, considering that $e(t) = y^o(t) - y(t)$, the state space model of the closed-loop system is given by

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_2(t) + y^o(t) \\ \dot{x}_2(t) &= 2x_1(t) - y^o(t) \\ y(t) &= -x_2(t) \end{aligned}$$

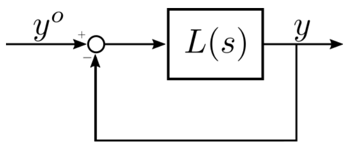
The state matrix of the closed-loop system is thus

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$$

whose eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$. As expected the closed-loop system has one pole in the right half plane that is the same pole that has been cancelled out in the computation of the loop transfer function.

Exercise 5

Given the following system



where

$$L(s) = \frac{10}{s(1+s)(1+0.02s)}$$

compute the step response time, and the steady-state error due to $y^o(t) = 1 + 2t$.

Solution

From the magnitude Bode diagram of the loop transfer function (Fig. 4) it follows that the crossover frequency is approximately equal to $3\,rad/s$. The phase at the crossover frequency is given by

$$\varphi_c = -90^o - \arctan(3) - \arctan(0.06) \approx -165^o$$

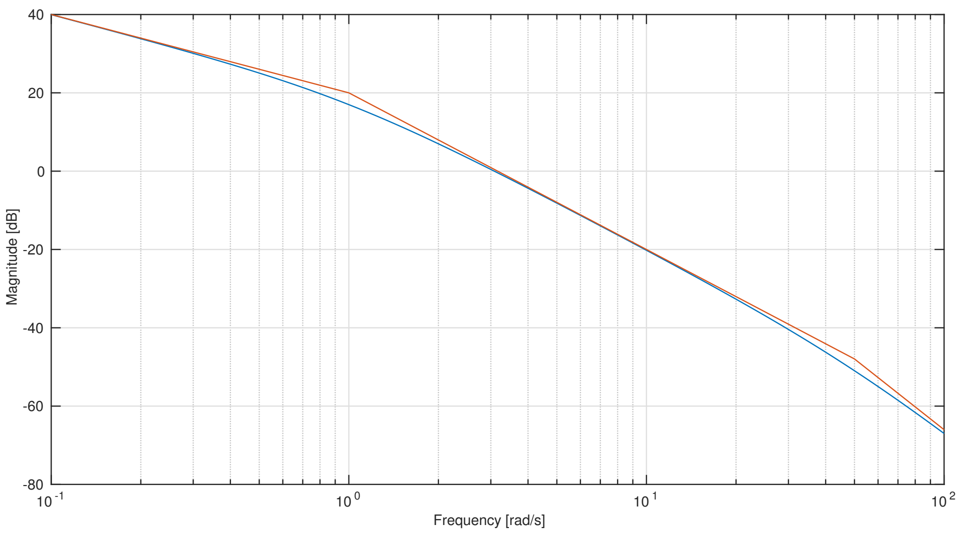


Figure 4: Loop transfer function Bode plot

As a consequence the phase margin is

$$\varphi_m = 180^o - \|\varphi_c\| \approx 15^o$$

and the system is asymptotically stable.
Due to the small phase margin the response will exhibit oscillations and the response time is given by

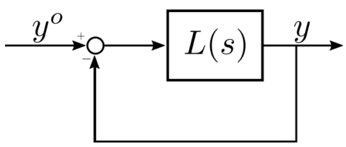
$$T_{a1} = \frac{5}{\xi \omega_c} \approx \frac{5}{3\frac{15}{100}} \approx 11.1\,s$$

The steady-state error due to $y^o(t)$ can be computed using the superimposition principle. Applying the final value theorem we get

$$e_\infty = \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{1}{s} + \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{2}{s^2} = \lim_{s\to 0}\frac{1}{1+\frac{10}{s}} + \lim_{s\to 0}\frac{1}{1+\frac{10}{s}}\frac{2}{s} = \lim_{s\to 0}\frac{s}{s+10} + \lim_{s\to 0}\frac{2}{s+10} = 0.2$$

Exercise 6

Given the following system



where

$$L(s) = \frac{10}{s^2}\frac{1+s}{1+0.01s}$$

assess the stability of the closed-loop system using the Bode criterion, and compute the step response time.

Solution

From the magnitude Bode diagram of the loop transfer function (Fig. 5) it follows that the crossover frequency is approximately equal to $10\,rad/s$. The phase at the crossover frequency is given by

$$\varphi_c = -180^o + \arctan(10) - \arctan(0.1) \approx -101.4^o$$

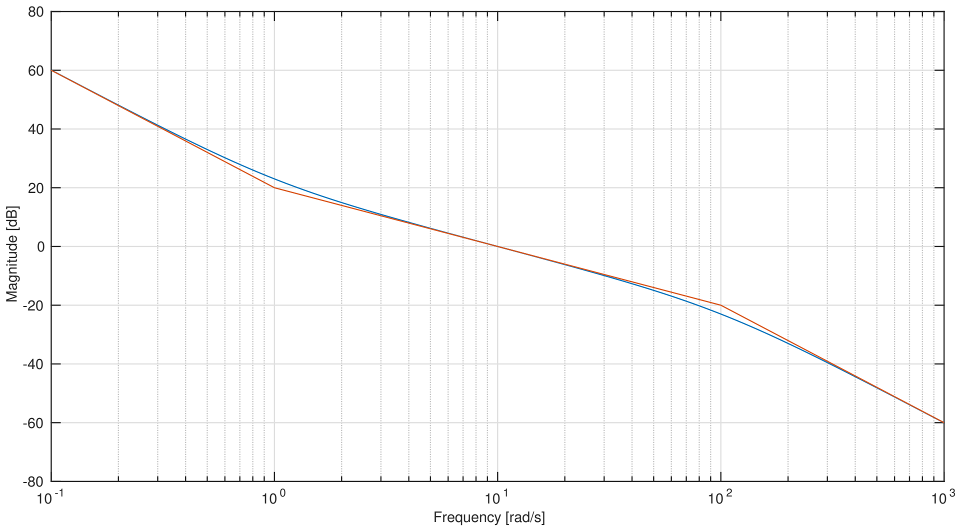


Figure 5: Loop transfer function Bode plot

As a consequence the phase margin is

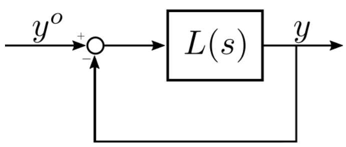
$$\varphi_m = 180^o - \|\varphi_c\| \approx 78.6^o$$

and the system is asymptotically stable.
Being the phase margin bigger than 60^o we can approximate the response time as follows

$$T_{a1} = \frac{5}{\omega_c} \approx \frac{5}{10} = 0.5\,s$$

Exercise 7

Given the following system



where

$$L(s) = \frac{10}{s^2 + s}$$

compute the steady-state error due to

- $y^o(t) = \text{sca}(t)$
- $y^o(t) = \text{ram}(t)$
- $y^o(t) = \text{sca}(t) + \text{par}(t)$

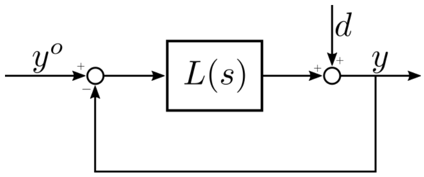
Solution

Applying the final value theorem we obtain

$$
\begin{aligned}
e_{\infty_{\mathrm{sca}(t)}} &= \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{1}{s} = \lim_{s\to 0}\frac{1}{1+\frac{10}{s}} = \lim_{s\to 0}\frac{s}{s+10} = 0\\
e_{\infty_{\mathrm{ram}(t)}} &= \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{1}{s^2} = \lim_{s\to 0}\frac{1}{1+\frac{10}{s}}\frac{1}{s} = \lim_{s\to 0}\frac{1}{s+10} = 0.1\\
e_{\infty_{\mathrm{par}(t)}} &= \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{1}{s^3} = \lim_{s\to 0}\frac{1}{1+\frac{10}{s}}\frac{1}{s^2} = \lim_{s\to 0}\frac{1}{s(s+10)} = +\infty\\
e_{\infty_{\mathrm{sca}(t)+\mathrm{par}(t)}} &= +\infty
\end{aligned}
$$

Exercise 8

Given the following system



where

$$L(s) = \mu\frac{1+10s}{(1+s)^2}$$

answer to the following questions

- compute the value of the loop gain μ in such a way that the cut off frequency of the closed-loop system is $1000\,rad/s$;
- using for μ the value previously computed, draw the response of the closed-loop system to a step on the reference signal (assuming $d(t) = 0$);
- find the range of frequencies for which the disturbance d is attenuated at least 10 times.

Solution

From the magnitude Bode diagram of the loop transfer function plotted for $\mu = 1$ (Fig. 6) it follows that, in order to have a crossover frequency approximately equal to $1000\,rad/s$, the digram has to be translated up of $40\,dB$ and this is equivalent to multiply the gain by $40\,dB = 100$. Consequently, $\mu = 100$.
With this value of μ the loop transfer function is

$$L(s) = 100\frac{1+10s}{(1+s)^2}$$

and the phase at the crossover frequency is given by

$$\varphi_c = \arctan(10000) - 2\arctan(1000) \approx -89.9^o$$

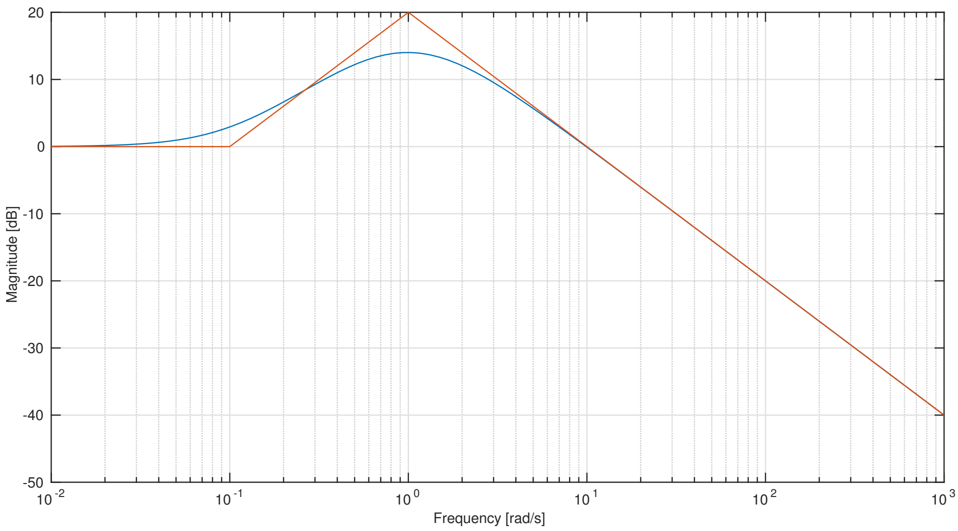


Figure 6: Loop transfer function Bode plot

As a consequence the phase margin is

$$\varphi_m = 180^o - \|\varphi_c\| \approx 90.1^o$$

and the system is asymptotically stable.
Being the phase margin bigger than 60^o we can approximate the complementary sensitivity function with the following first order system

$$F(s) \approx \frac{\tilde{\mu}}{1 + 0.\frac{s}{\omega_c}} = \frac{\tilde{\mu}}{1 + 0.001s}$$

and the response time with

$$T_{a1} = \frac{5}{\omega_c} \approx \frac{5}{1000} = 0.005\, s$$

The gain $\tilde{\mu}$ of the complementary sensitivity function is very close to one, but it is not exactly one as the type of the loop transfer function is zero. We can compute the value of the gain using the final value theorem

$$y_\infty = \lim_{s \to 0} s \frac{L(s)}{1 + L(s)} \frac{1}{s} = \lim_{s \to 0} \frac{100}{1 + 100} = \frac{100}{101}$$

The step response is shown in Fig. 7.

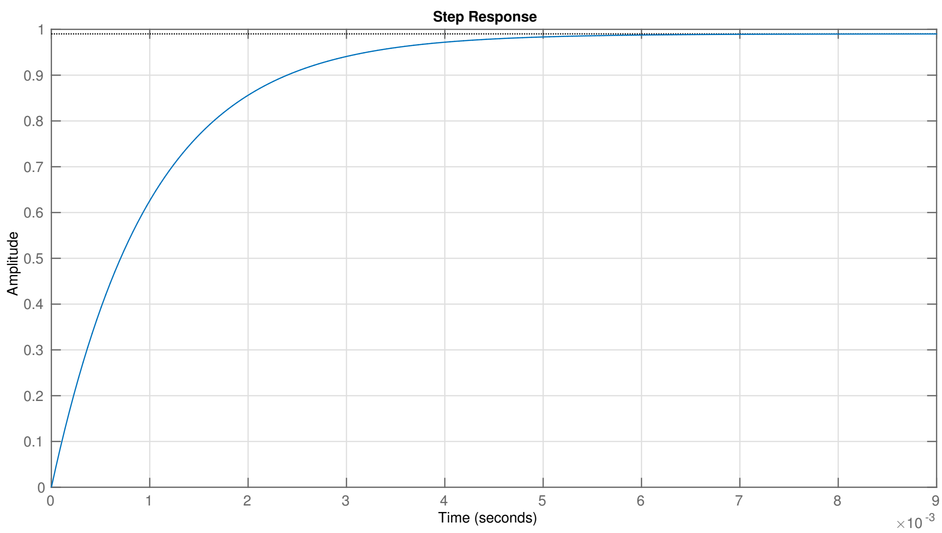


Figure 7: Step response of the approximated complementary sensitivity function

The magnitude of the frequency response associated to the transfer function from the disturbance d to the output y is given by

$$\left|\frac{Y(j\omega)}{D(j\omega)}\right| = \left|\frac{1}{1+L(j\omega)}\right| \approx \begin{cases} \frac{1}{|L(j\omega)|} & \omega \leq \omega_c \\ 1 & \omega > \omega_c \end{cases}$$

and its Bode diagram can be obtained directly from the magnitude Bode diagram of the loop transfer function (Fig. 8).

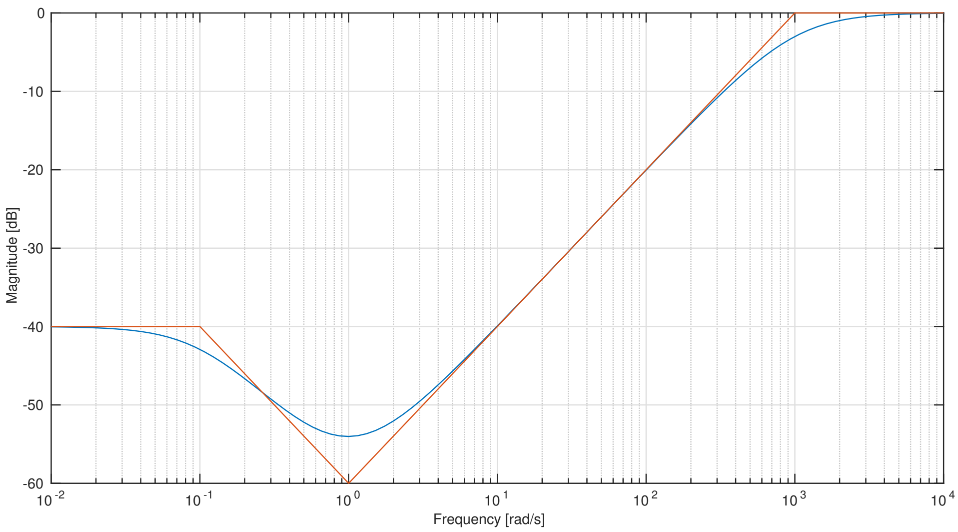
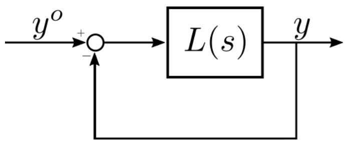


Figure 8: Approximated magnitude Bode diagram of the sensitivity function

Considering that the amplification/attenuation operated by the closed-loop system on the disturbance is given by the magnitude of the loop sensitivity function, from Fig. 8 it follows that the disturbance is attenuated of at least 10 times in the range of frequencies $[0, 100]$ rad/s.

Exercise 9

Given the following system



where

$$L(s) = \frac{10}{s(1+0.02s)}$$

Is the system able to track the following reference signal

$$y^o(t) = a_1 \sin(t+b_1) + a_2 \sin(5t+b_2) + a_3 \sin(100t+b_3)$$

where a_1, a_2, a_3, b_1, b_2, b_3 are general real values.

Solution

From the magnitude Bode diagram of the loop transfer function (Fig. 9) it follows that the crossover frequency is approximately equal to $10\,rad/s$.

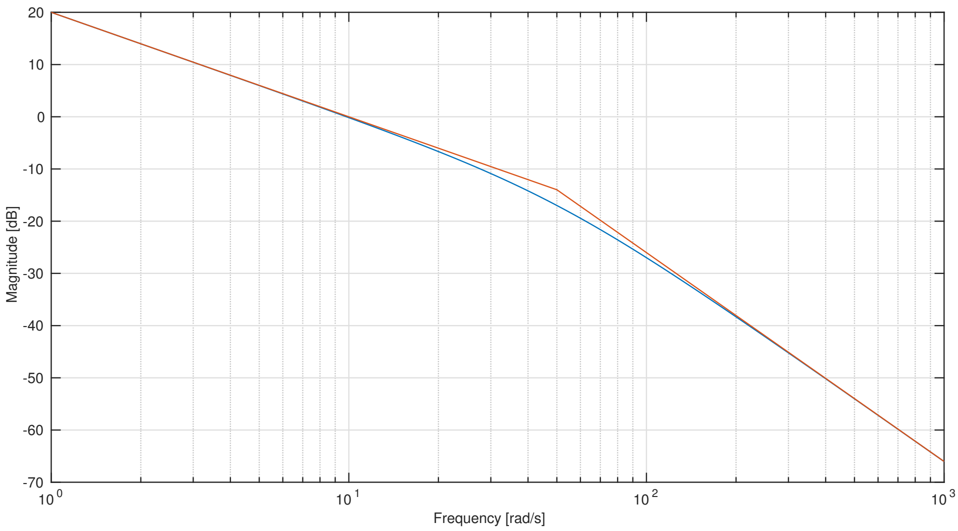
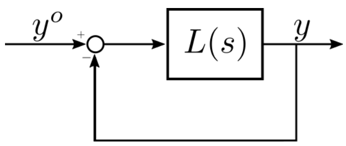


Figure 9: Loop transfer function Bode plot

As a consequence, the third harmonics of the reference cannot be reproduced on the output as its frequency is greater than the crossover frequency.

Exercise 10

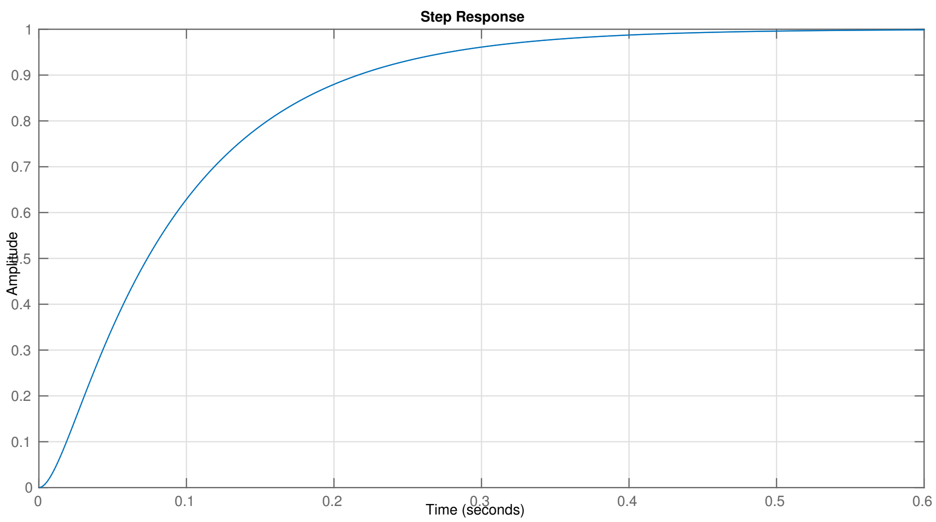
Given the following system



Which of the following transfer functions

$$L_1(s) = \frac{10}{s(1+0.01s)} \quad L_2(s) = \frac{1}{s(1+0.1s)} \quad L_3(s) = \frac{10}{(1+s)(1+0.01s)} \quad L_4(s) = \frac{10}{1+0.1s}$$

have the following response to a unitary reference step signal?



Solution

From the step response we can observe that:

- the shown step response is very close to the step response of a first order approximation of the closed-loop system;
- $y_\infty = 1$ and thus $e_\infty = 0$, as a consequence the loop transfer function has at least type 1;
- the transient ends after $0.5\,s$, as a consequence the crossover frequency is approximately equal to $10\,rad/s$.

As a consequence, $L_3(s)$ and $L_4(s)$ cannot be the loop transfer function of the system, as they have type 0. Moreover, as it follows from Fig. 10 the crossover frequency associated to $L_2(s)$ is $1\,rad/s$ and is thus non compatible with the characteristics of the given step response.

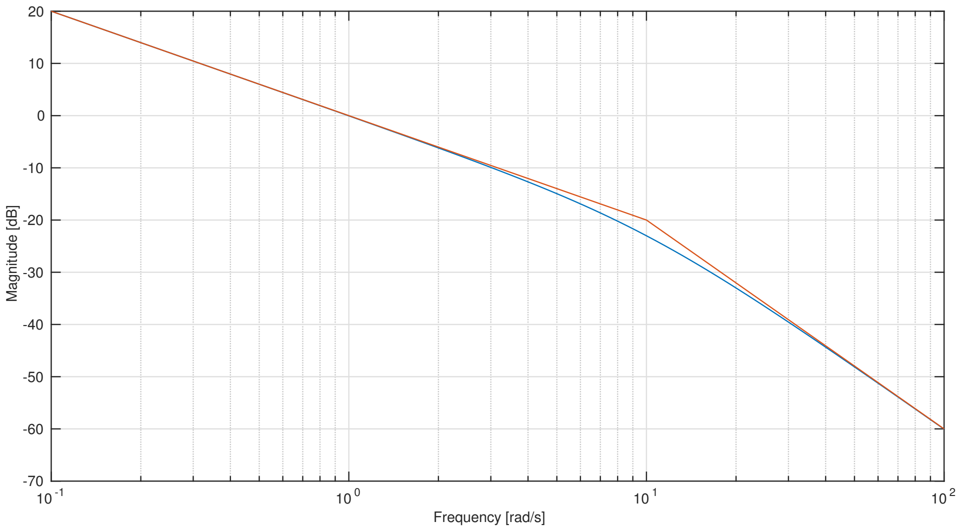


Figure 10: Loop transfer function Bode plot

In conclusion, $L_1(s)$ has the step response shown in the figure.