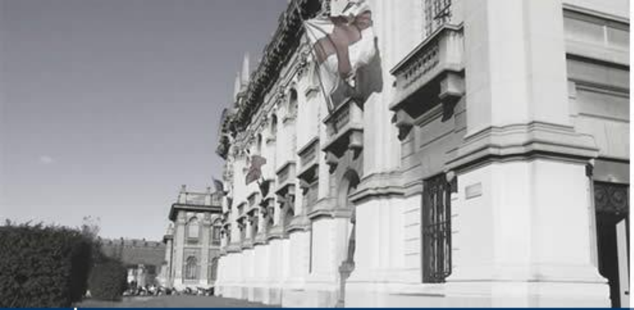

POLITECNICO DI MILANO

Automatic Control

Systems theory overview (discrete time systems)

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Motivations

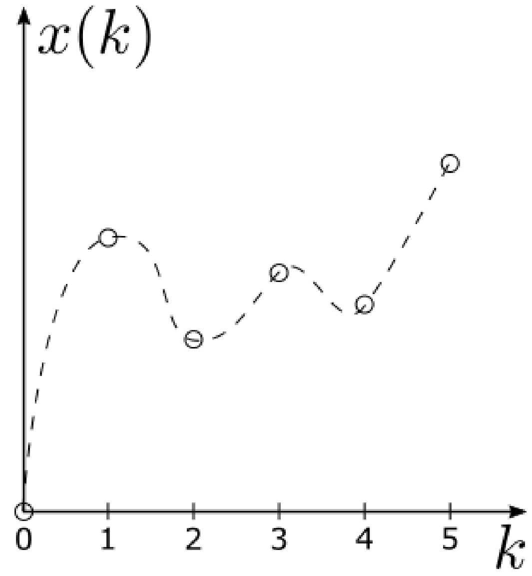
We complete the fundamentals of systems theory with the basic knowledge on discrete time systems.

The main topics we will face are:

- discrete time systems
- Linear Time Invariant systems
- stability of equilibria of nonlinear systems
- the Z transform
- transfer function of a LTI system
- time response of a first order system
- frequency response

Discrete time systems (I)

Discrete time systems are dynamical systems whose variables are referred to a "time" that is not continuous but discrete, i.e. "time" k is an integer number.



Why are we interested to consider discrete time systems?

- "Natura non facit saltus", but there are systems in economics, ecology, sociology, etc. that can be naturally described with discrete time systems
- the control algorithm executed by a (embedded) processor evolves like a discrete time system

Discrete time systems (II)

A discrete time system is characterized by m input and p output variables.

As in continuous time systems, we call order n of the dynamical system the minimum number of initial conditions we need to compute the system output given the input values from the initial time.

A discrete time system is described by the following n state and p output difference equations

$$\mathbf{x}(k+1) = \mathbf{f}\left(\mathbf{x}(k), \mathbf{u}(k), k\right)$$

$$\mathbf{y}(k) = \mathbf{g}\left(\mathbf{x}(k), \mathbf{u}(k), k\right)$$

We could introduce the same classification presented for continuous time systems. Discrete time systems can be: SISO or MIMO, strictly proper or proper, linear or nonlinear, time invariant or time varying.

Discrete time systems (III)

Given a discrete time system, an initial condition at time k_0, and an input function for $k \geq k_0$, we call

- state trajectory, a solution of the state equations that starts from the given initial condition
- output trajectory, the trajectory determined by the output equations given the state trajectory

A constant trajectory, generated by a constant input function, is called equilibrium.

Given a constant input $\bar{u}$, the equilibria are solutions of the following equations

$$\bar{\mathbf{x}} = \mathbf{f}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$$

$$\bar{\mathbf{y}} = \mathbf{g}(\bar{\mathbf{x}}, \bar{\mathbf{u}})$$

We could introduce the same definitions of stable, unstable, and asymptotically stable trajectory (or equilibrium).

Discrete time systems – Example

We would like to find an algorithm to compute the solutions of the scalar equation

$$z = f(z)$$

where f is a general nonlinear function.

We start from a guess solution x_0 and iterate following the algorithm

$$x(k+1) = f(x(k))$$

$$x(0) = x_0$$

where k represent the iteration index.

We observe that the equilibria of the previous system are the solutions of the equation

$$\bar{x} = f(\bar{x})$$

i.e., the equilibria are the solutions we are looking for.

We conclude that, if the equilibria are asymptotically stable and the guess solution is sufficiently close to them, after some iterations the algorithm will converge to the solutions we are looking for.

If all the functions f_i and g_i are linear with respect to the state and input variables, and do not depend on time k, the discrete time system is called Linear and Time Invariant (LTI) system.

A LTI discrete time system is described by the following equations

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

Given an initial condition $\boldsymbol{x}_0$, we can iteratively compute the state trajectory

$$\mathbf{x}(1) = \mathbf{A}\mathbf{x}(0) + \mathbf{B}\mathbf{u}(0) = \mathbf{A}\mathbf{x}_0 + \mathbf{B}\mathbf{u}(0)$$
$$\mathbf{x}(2) = \mathbf{A}\mathbf{x}(1) + \mathbf{B}\mathbf{u}(1) = \mathbf{A}^2\mathbf{x}_0 + \mathbf{A}\mathbf{B}\mathbf{u}(0) + \mathbf{B}\mathbf{u}(1)$$
$$\mathbf{x}(3) = \mathbf{A}\mathbf{x}(2) + \mathbf{B}\mathbf{u}(2) = \mathbf{A}^3\mathbf{x}_0 + \mathbf{A}^2\mathbf{B}\mathbf{u}(0) + \mathbf{A}\mathbf{B}\mathbf{u}(1) + \mathbf{B}\mathbf{u}(2)$$
$$\vdots$$

From these relations we can derive the general expression of the state trajectory of LTI discrete time systems.

Linear Time Invariant systems (II)

$$\mathbf{x}(k) = \boxed{\mathbf{A}^k \mathbf{x}_0} + \boxed{\sum_{i=0}^{k-1} \left[\mathbf{A}^{k-i-1}\mathbf{B}\mathbf{u}(i)\right]}$$

$$\mathbf{y}(k) = \boxed{\mathbf{C}\mathbf{A}^k \mathbf{x}_0} + \boxed{\sum_{i=0}^{k-1} \left[\mathbf{C}\mathbf{A}^{k-i-1}\mathbf{B}\mathbf{u}(i)\right] + \mathbf{D}\mathbf{u}(k)}$$

<u>Zero-input response</u>
Generated by the initial condition only

<u>Zero-state response</u>
Generated by the input only

Observing that

- the zero-input response is linear with respect to the initial condition
- the zero-state response is linear with respect to the input

we conclude that for LTI discrete time systems the <u>superposition principle</u> holds.

Given a LTI discrete time system

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

the state equilibria are the solutions of the following equation

$$\bar{\mathbf{x}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{B}\bar{\mathbf{u}}$$

If matrix $\mathbf{I_n} - \mathbf{A}$ is non singular (i.e., $\mathbf{A}$ has no eigenvalues $\lambda_i = 1$), there exists a unique state equilibrium given by

$$\bar{\mathbf{x}} = (\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\bar{\mathbf{u}}$$

and the output equilibrium is

$$\bar{\mathbf{y}} = \boxed{\left[\mathbf{C}(\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]}\bar{\mathbf{u}}$$

Static gain

We will now recall other properties of continuous time systems that hold for discrete time systems as well.

Change of variables

$$\hat{\mathbf{x}}(k) = \mathbf{T}\mathbf{x}(k) \qquad \det(\mathbf{T}) \neq 0$$

$$\begin{aligned} \hat{\mathbf{x}}(k+1) &= \hat{\mathbf{A}}\hat{\mathbf{x}}(k) + \hat{\mathbf{B}}\mathbf{u}(k) \\ \hat{\mathbf{y}}(k) &= \hat{\mathbf{C}}\hat{\mathbf{x}}(k) + \hat{\mathbf{D}}\mathbf{u}(k) \end{aligned} \qquad \begin{aligned} \hat{\mathbf{A}} &= \mathbf{T}\mathbf{A}\mathbf{T}^{-1} & \hat{\mathbf{B}} &= \mathbf{T}\mathbf{B} \\ \hat{\mathbf{C}} &= \mathbf{C}\mathbf{T}^{-1} & \hat{\mathbf{D}} &= \mathbf{D} \end{aligned}$$

Controllability

$$\mathbf{K}_r = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

The system is completely controllable if and only if $\mathrm{rank}(\mathbf{K}_r) = n$.

Observability

$$\mathbf{K}_o = \begin{bmatrix} \mathbf{C}^T & \mathbf{A}^T\mathbf{C}^T & \mathbf{A}^{T^2}\mathbf{C}^T & \cdots & \mathbf{A}^{T^{n-1}}\mathbf{C}^T \end{bmatrix}$$

The system is completely observable if and only if $\mathrm{rank}(\mathbf{K}_o) = n$.

Linear Time Invariant systems (V)

Concerning the stability of LTI discrete time systems, the following conclusions, already derived for continuous time systems, hold

- the trajectories are all stable, all unstable or all asymptotically stable
- stability is a property of the system
- stability can be assessed studying the zero-input response of the system

Analyzing the stability of continuous time systems, we discovered it depends on the zero-input response of the following system

$$\delta \mathbf{x}(k+1) = \mathbf{A}\delta \mathbf{x}(k)$$

where $\delta \mathbf{x}$ is the difference between the nominal and the perturbed state trajectory.

The zero-input response is given by

$$\delta \mathbf{x}(k) = \mathbf{A}^k \delta \mathbf{x}(0)$$

Assuming the state matrix $\mathbf{A}$ is diagonalizable, we can introduce a change of variables that decouples the trajectories and simplifies the computation of the matrix exponential.

Linear Time Invariant systems (VI)

$$\delta\mathbf{x}(k) = \mathbf{A}^k\delta\mathbf{x}_0 = \left(\mathbf{T}^{-1}\hat{\mathbf{A}}\mathbf{T}\right)^k\delta\mathbf{x}_0$$
$$= \mathbf{T}^{-1}\hat{\mathbf{A}}^k\mathbf{T}\delta\mathbf{x}_0 = \mathbf{T}^{-1}\mathrm{diag}\left(\lambda_1^k,\ldots,\lambda_n^k\right)\mathbf{T}\delta\mathbf{x}_0$$

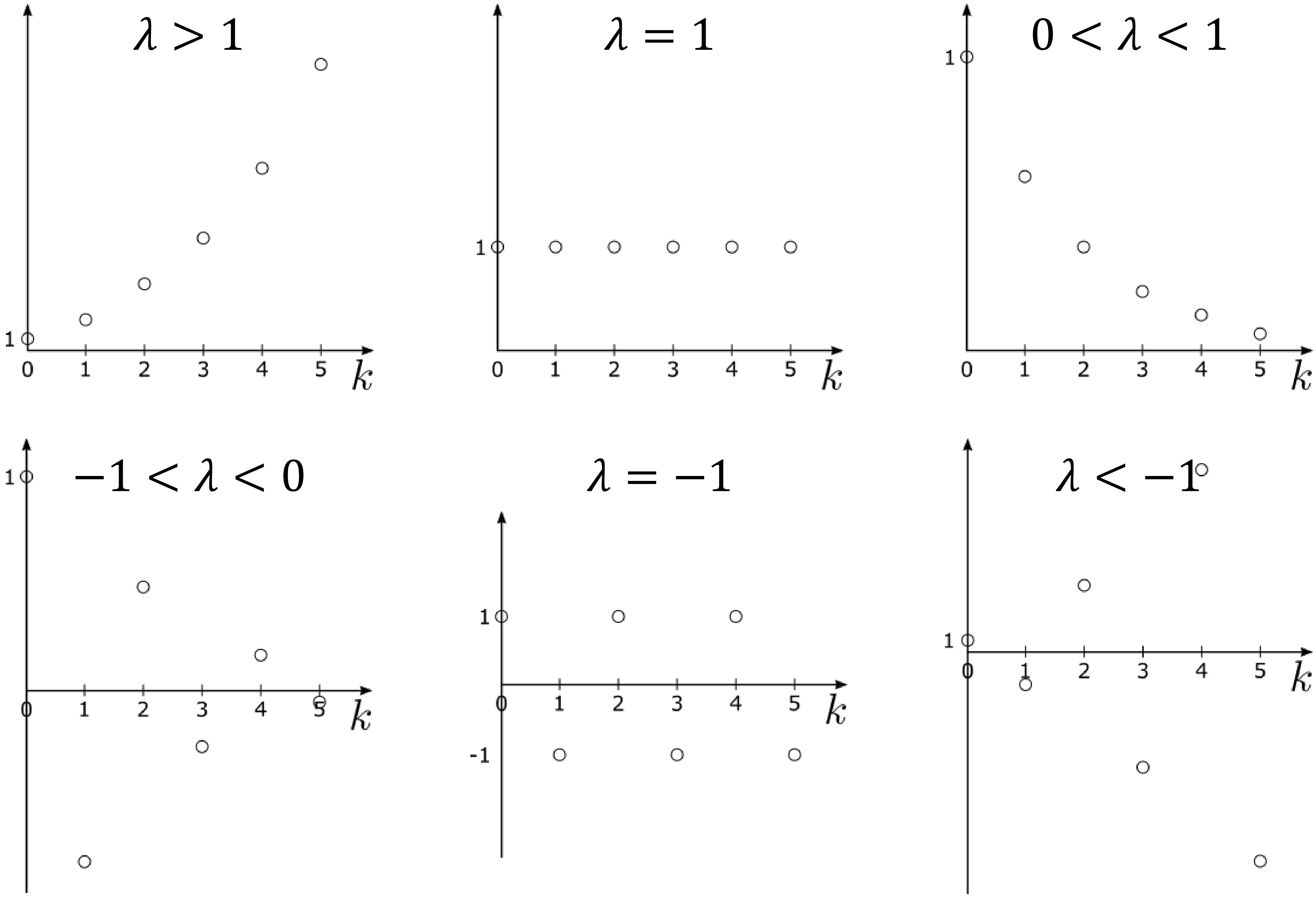
We conclude that the zero-input responses are linear combinations of the terms

$$\lambda_i^k$$

that we call <u>characteristic modes</u> or natural modes of the LTI discrete time system.

Let's look at the different behavior of the modes for $\lambda_i \in \mathbb{R}$.

Linear Time Invariant systems (VII)



Linear Time Invariant systems (VIII)

Let's analyze the modes, assuming that $\lambda_i \in \mathbb{C}$ $(\lambda_i = \rho_i e^{j\theta_i})$

$$\lambda_i^k = \rho_i^k e^{jk\theta_i} = \rho_i^k \left(\cos(\theta_i k) + j\sin(\theta_i k)\right)$$

making the linear combination, the imaginary part is cancelled out by the imaginary part of the complex conjugate of λ_i.

As a consequence we have

- λ_i^k when $\lambda_i \in \mathbb{R}$
- $\rho_i^k \cos(\theta_i k + \varphi_i)$ when $\lambda_i \in \mathbb{C}$

Let's know analyze these two modes in view of the stability condition.

The analysis of the two modes λ_i^k and $\rho_i^k \cos(\theta_i k + \varphi_i)$ reveals that:

- if all the eigenvalues of matrix $\mathbf{A}$ lie inside the unit circle ($\rho_i < 1$), all the modes are bounded and tend to zero asymptotically
- if all the eigenvalues of matrix $\mathbf{A}$ lie inside the unit circle or on the circumference ($\rho_i \leq 1$), and there is at least one eigenvalue on the circumference ($\rho_i = 1$), all the modes are bounded but the modes associated to the eigenvalues on the circumference do not tend to zero asymptotically
- if at least one eigenvalue of matrix $\mathbf{A}$ lies outside the unit circle ($\rho_i > 1$), there is at least one mode that is not bounded

Based on the previous analysis we conclude that an LTI discrete time system with diagonalizable state matrix is:

- asymptotically stable, if and only if all the eigenvalues of matrix $\mathbf{A}$ lie inside the unit circle ($|\lambda_i| < 1\ \forall i$)
- stable, if and only if all the eigenvalues of matrix $\mathbf{A}$ lie inside the unit circle ($|\lambda_i| \leq 1\ \forall i$) and there is at least one eigenvalue on the circumference ($\exists i{:}\ |\lambda_i| = 1$)
- unstable, if and only if there is at least one eigenvalue of matrix $\mathbf{A}$ lying outside the unit circle ($\exists i{:}\ |\lambda_i| > 1$)

In the general case of non diagonalizable state matrices it can be shown that, if all the eigenvalues of matrix $\mathbf{A}$ lie inside the unit circle, and there are multiple eigenvalues on the unit circle, the system is unstable if there is at least one eigenvalue on the unit circle whose geometric multiplicity is less than the algebraic multiplicity.

As for continuous time systems, the stability is a structural property of the system.

Linear Time Invariant systems (XI)

As we did for continuous time systems, we would like to investigate the existence of tools to perform the stability analysis without computing the eigenvalues.

If we focused on criteria to assess the stability of a system analyzing its characteristic polynomial, we can proceed in two ways:

- Jury criterion, it gives the conditions to ensure that a polynomial has all the roots inside the unit circle
- introduce a change of variables ensuring that, if the transformed polynomial has all the roots in the open left half plane, the original polynomial has all the roots inside the unit circle

An example of change of variables satisfying this property is the bilinear transformation

$$z = \frac{1+s}{1-s}$$

Consider the following characteristic polynomial

$$\varphi(z) = 8z^3 - 12z^2 + 6z - 1$$

applying the bilinear transformation we obtain

$$\varphi(z)\big|_{z=\left(\frac{1+s}{1-s}\right)} = 8\left(\frac{1+s}{1-s}\right)^3 - 12\left(\frac{1+s}{1-s}\right)^2 + 6\left(\frac{1+s}{1-s}\right) - 1$$

Equating to zero in order to create the characteristic equation

$$8(1+s)^3 - 12(1+s)^2(1-s) + 6(1+s)(1-s)^2 - (1-s)^3 = 0$$

and expanding the powers

$$27s^3 + 27s^2 + 9s + 1 = 0$$

To this polynomial we can apply the Routh criterion

$$\begin{array}{ccc} 27 & 9 & 0 \\ 27 & 1 & 0 \\ 8 & 0 & \\ 1 & & \end{array}$$

We conclude that the polynomial in “s” has all the roots in the open left half plane and, consequently, the polynomial in “z” has all the roots inside the unit circle.

Given a nonlinear time invariant system

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)) \qquad \mathbf{x} \in \mathbb{R}^n \quad \mathbf{u} \in \mathbb{R}^m$$

$$\mathbf{y}(k) = \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k)) \qquad \mathbf{y} \in \mathbb{R}^p$$

and an equilibrium

$$\mathbf{u}(k) = \bar{\mathbf{u}} \quad \mathbf{x}(k) = \bar{\mathbf{x}} \quad \mathbf{y}(k) = \bar{\mathbf{y}} \quad \forall k$$

We can locally approximate the nonlinear system, around the equilibrium, with the linearized system

$$\delta\mathbf{x}(k+1) = \mathbf{A}\delta\mathbf{x}(k) + \mathbf{B}\delta\mathbf{u}(k)$$

$$\delta\mathbf{y}(k) = \mathbf{C}\delta\mathbf{x}(k) + \mathbf{D}\delta\mathbf{u}(k)$$

where

$$\delta\mathbf{x}(k) = \mathbf{x}(k) - \bar{\mathbf{x}} \quad \delta\mathbf{u}(k) = \mathbf{u}(k) - \bar{\mathbf{u}} \quad \delta\mathbf{y}(k) = \mathbf{y}(k) - \bar{\mathbf{y}}$$

and

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \quad \mathbf{B} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \quad \mathbf{C} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}} \quad \mathbf{D} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{u}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$$

Stability of equilibria of nonlinear systems (II)

As the linearized system is a LTI system, we can assess the stability of the equilibrium point of the nonlinear system analyzing the state matrix

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}},\bar{\mathbf{u}}}$$

We can state the following results:

- if all the eigenvalues of matrix $\mathbf{A}$ lie inside the unit circle ($|\lambda_i| < 1$), <u>the equilibrium point is asymptotically stable</u>
- if at least one eigenvalue of matrix $\mathbf{A}$ lies outside the unit circle ($\exists i: |\lambda_i| > 1$), <u>the equilibrium point is unstable</u>

If the eigenvalues of matrix $\mathbf{A}$ lie inside the unit circle and there is at least one eigenvalue on the circumference the linearization, that is a first order approximation, is too rough to assess the stability of the equilibrium point.

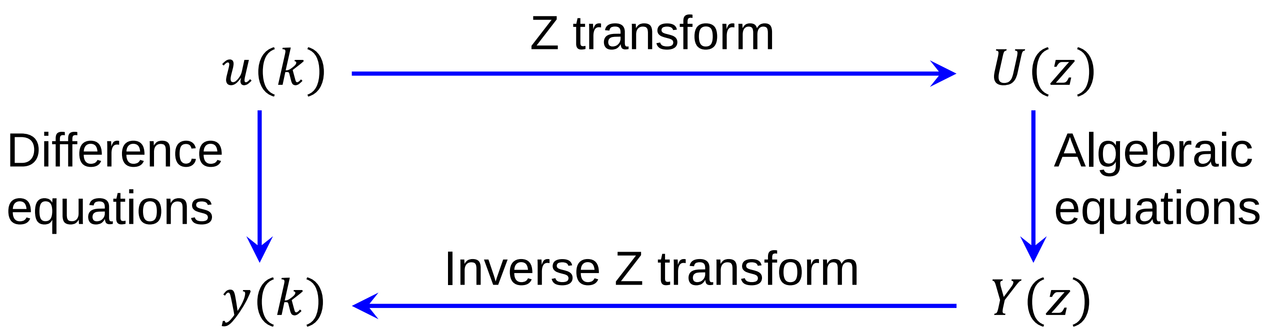
The Z transform (I)

Given a LTI system

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

as we did for continuous time systems, we can introduce a representation of the LTI system in the frequency domain



The input-output relation in the frequency domain is again an algebraic relation between the input and output transforms.

For discrete time systems this transformation is performed using the Z transform.

The Z transform (II)

Given a real function $v(k)$, where k is an integer number and $k \geq 0$, we call Z transform of v the complex function

$$V(z) = \sum_{k=0}^{\infty} v(k) z^{-k}$$

of the complex variable z.

Usually this series converges only for the values of z lying outside a circle of radius r ($|z| > r > 0$).

We will assume as the Z transform of $v(k)$ the sum of the series computed for the values of z where the series is convergent.

Let's see how to compute the Z transform of some common signals.

Discrete time unitary impulse

Given the unitary impulse (Kronecker delta)

$$v(k) = \mathrm{imp}(k) = \delta_0(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

we have

$$V(z) = \sum_{k=0}^{\infty} v(k) z^{-k} = v(0) + v(1)z^{-1} + v(2)z^{-2} + \cdots = 1 + 0 + 0 + \cdots = 1$$

Discrete time exponential

Given a discrete time exponential $v(k) = a^k$ we have

$$V(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \sum_{k=0}^{\infty} \left(az^{-1}\right)^k = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \qquad (|z| > |a|)$$

If $a = 1$ then $v(k) = \mathrm{sca}(k)$ and the Z transform is

$$V(z) = \frac{z}{z-1}$$

The Z transform (IV)

We now review the most important properties of the Z transform.

Linearity

$$v(k) = \alpha v_1(k) + \beta v_2(k) \quad \Rightarrow \quad V(z) = \alpha V_1(z) + \beta V_2(z)$$

Time shift

$$v_2(k) = v_1(k+1) \quad \Rightarrow \quad V_2(z) = z\left(V_1(z) - v_1(0)\right)$$

$$v_2(k) = v_1(k-1) \quad \Rightarrow \quad V_2(z) = z^{-1} V_1(z)$$

First derivative in z-domain

$$v_2(k) = k v_1(k) \quad \Rightarrow \quad V_2(z) = -z \frac{\mathrm{d}V_1(z)}{\mathrm{d}z}$$

Initial value theorem

$$v(0) = \lim_{z \to \infty} V(z)$$

Final value theorem (if $|p_i| < 1$ or $p_i = 1$)

$$v(\infty) = \lim_{k \to \infty} v(k) = \lim_{z \to 1} \left[(z-1)V(z)\right]$$

The Z transform – Examples

Discrete time ramp

Given the ramp signal

$$\mathrm{ram}(k) = k \quad k \geq 0$$

we have

$$\mathscr{Z}\left[\mathrm{ram}(k)\right] = -z\frac{\mathrm{d}}{\mathrm{d}z}\left\{\mathscr{Z}\left[\mathrm{sca}(k)\right]\right\} = -z\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{z}{z-1}\right] = \frac{z}{(z-1)^2}$$

Discrete time exponential

Consider a signal with Z transform $V(z) = \frac{z}{z-a}$, from the initial and final value theorems we have

The results are coherent with $v(k) = a^k$

$$v(0) = \lim_{z\to\infty} V(z) = 1$$

$$|a| < 1 \quad \Rightarrow \quad v(\infty) = \lim_{z\to 1}\left[(z-1)V(z)\right] = \lim_{z\to 1}\left[(z-1)\frac{z}{z-a}\right] = 0$$

$$a = 1 \quad \Rightarrow \quad v(\infty) = \lim_{z\to 1}\left[(z-1)V(z)\right] = \lim_{z\to 1}\left[(z-1)\frac{z}{z-1}\right] = 1$$

Let's recap the Z transform of the main signals we will face studying discrete time systems.

$v(k)$	$V(z)$
$\mathrm{imp}(k)$	1
$\mathrm{sca}(k)$	$\frac{z}{z-1}$
$\mathrm{ram}(k)$	$\frac{z}{(z-1)^2}$
$\mathrm{par}(k)$	$\frac{z}{(z-1)^3}$
a^k	$\frac{z}{(z-a)}$
ka^k	$\frac{az}{(z-a)^2}$

$$\mathrm{par}(k) = \frac{k(k-1)}{2} \qquad k \geq 0$$

The Z transform (VI)

In order to come back to the time domain we need to introduce the inverse transform.

If we focused on rational Z transforms (ratio of polynomials), we can use the Heaviside method to expand the rational function in partial fractions.

With Z transform is better to expand $\frac{V(z)}{z}$ instead of $V(z)$.

Let's see the example of a Z transform with distinct roots.

The expansion in partial fractions is given by

$$\frac{V(z)}{z} = \frac{\alpha_0}{z} + \frac{\alpha_1}{z - p_1} + \cdots + \frac{\alpha_n}{z - p_n}$$

multiplying now each side by z

$$V(z) = \alpha_0 + \frac{\alpha_1 z}{z - p_1} + \cdots + \frac{\alpha_n z}{z - p_n}$$

and applying the inverse transform at each addend

$$v(k) = \alpha_0 \mathrm{imp}(k) + \alpha_1 p_1^k + \cdots + \alpha_n p_n^k \quad k \geq 0$$

The Z transform (VII)

Another technique to recover the signal in the time domain is the polynomial long division, that allows to compute the samples of the time domain signal.

Consider the ratio between numerator and denominator

$$V(z) = \frac{N(z)}{D(z)} = \beta_0 + \beta_1 z^{-1} + \beta_2 z^{-2} + \dots$$

from this expression the samples of $v(k)$ follow

$$v(0) = \beta_0 \quad v(1) = \beta_1 \quad v(2) = \beta_2$$

The Z transform – Example (I)

Consider the following Z transform

$$V(z) = \frac{3z+12}{z^2+5z+6}$$

Applying Heaviside method we obtain

$$\frac{V(z)}{z} = \frac{3z+12}{z(z+2)(z+3)} = \frac{\alpha_0}{z} + \frac{\alpha_1}{z+2} + \frac{\alpha_2}{z+3}$$

$$= \frac{\alpha_0(z+2)(z+3) + \alpha_1 z(z+3) + \alpha_2 z(z+2)}{z(z+2)(z+3)}$$

and evaluating the numerator for $z = 0$, $z = -2$, $z = -3$

$$\begin{cases} 6\alpha_0 = 12 \\ -2\alpha_1 = 6 \\ 3\alpha_2 = 3 \end{cases} \Rightarrow \begin{cases} \alpha_0 = 2 \\ \alpha_1 = -3 \\ \alpha_2 = 1 \end{cases}$$

Consequently

$$V(z) = 2 - 3\frac{z}{z+2} + \frac{z}{z+3}$$

and applying the inverse transform

$$v(k) = 2\mathrm{imp}(k) - 3(-2)^k + (-3)^k$$

What happens if we perform the long division?

$$\begin{array}{l|l} 3z+12 & z^2+5z+6 \\ \hline 3z+15+18z^{-1} & 3z^{-1}-3z^{-2}-3z^{-3} \\ \hline -3-18z^{-1} & \\ -3-15z^{-1}-18z^{-2} & \\ \hline -3z^{-1}+18z^{-2} & \end{array}$$

The first four samples are

$$v(0) = 0 \quad v(1) = 3 \quad v(2) = -3 \quad v(3) = -3$$

and are coherent with the previous expression of $v(k)$.

Given a LTI system

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k)$$

and assuming zero initial conditions, we apply the Z transform to both sides of the equations, obtaining

$$\begin{aligned} z\mathbf{X}(z) &= \mathbf{A}\mathbf{X}(z) + \mathbf{B}\mathbf{U}(z) \\ \mathbf{Y}(z) &= \mathbf{C}\mathbf{X}(z) + \mathbf{D}\mathbf{U}(z) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{X}(z) &= (z\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(z) \\ \mathbf{Y}(z) &= \left[\mathbf{C}(z\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(z) \end{aligned}$$

The relation in the frequency domain between $\mathbf{U}(z)$ and $\mathbf{Y}(z)$ is called transfer function and it is given by

$$\mathbf{G}(z) = \mathbf{C}(z\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

We observe that $\mathbf{G}(z)$ is a $p \ x \ m$ matrix.

Transfer function of a LTI system (II)

The transfer function for a discrete time and for a continuous time system has the same analytical expression.

For this reason they also shares the same properties:

- the transfer function is invariant with respect to change of variables
- for SISO systems the transfer function is a ratio between two polynomials

$$G(z) = \frac{N(z)}{D(z)}$$

- we call zeros the roots of the numerator, and poles the roots of the denominator
- if there are no zero-pole cancellations the poles coincide with the eigenvalues of matrix $\mathbf{A}$

Transfer function of a LTI system (III)

We call type g of a transfer function the number of poles/zeros in $z = 1$.

As a consequence if:

- $g \geq 1$, there are g poles in $z = 1$
- $g = 0$, there are no zeros/poles in $z = 1$
- $g \leq -1$, there are $-g$ zeros in $z = 1$

Given a type 0 transfer function, we call the following constant

$$\mu = G(1) = \mathbf{C}\left(\mathbf{I}_n - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$$

gain of the transfer function.

We observer that in this case ($g = 0$) the gain of the transfer function is equal to the static gain of the LTI system.

If $g \neq 0$ the gain definition can be generalized as follows

$$\mu = \lim_{z \to 1}\left[(z-1)^g G(z)\right]$$

Transfer function of a LTI system (IV)

We conclude observing that in the case of discrete time systems the delay has a rational transfer function.

Consider a time delay of h discrete steps

$$y(k) = u(k-h)$$

Transforming each side of this relation we obtain

$$Y(z) = z^{-h}U(z)$$

and the transfer function of the time delay is

$$G(z) = z^{-h} = \frac{1}{z^h}$$

a unity gain system with h poles in $z = 0$.

Time response of a first order system (I)

Consider a first order system

$$G(z) = \mu \frac{1-p}{z-p}$$

We can compute the step response

$$u(k) = \mathrm{sca}(k) \quad \Rightarrow \quad U(z) = \frac{z}{z-1}$$

and

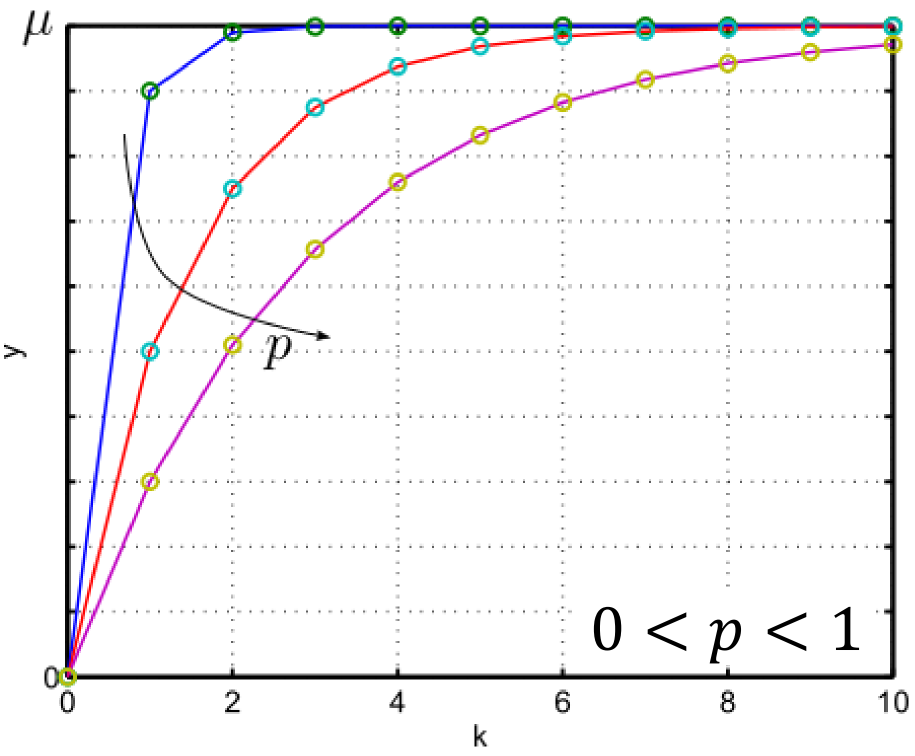
$$Y(z) = \mu \frac{1-p}{z-p} \frac{z}{z-1} = \mu \left(\frac{z}{z-1} - \frac{z}{z-p} \right)$$

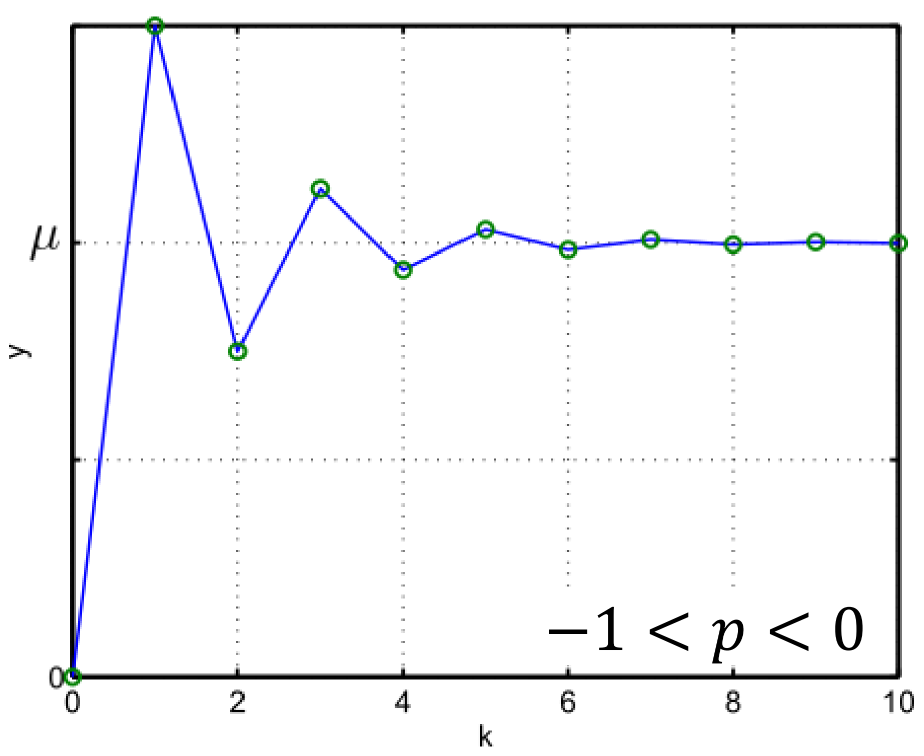
Applying the inverse Z transform we obtain

$$y(k) = \mu \left(1 - p^k \right) \quad k \geq 0$$

If $|p| < 1$ the system is asymptotically stable and the step response asymptotically converges to μ.

Time response of a first order system (II)





A peculiarity of discrete time systems: even the step response of a first order system can exhibit oscillations.

Frequency response (I)

Given a general LTI asymptotically stable discrete time system, represented by the transfer function $G(z)$, in steady state a sinusoidal input

$$u(k) = U \sin\left(\bar{\theta}k + \phi\right)$$

generates a sinusoidal response of the same frequency

$$y(k) = Y \sin\left(\bar{\theta}k + \psi\right)$$

$$Y = U \left| G\left(\mathrm{e}^{\mathrm{j}\bar{\theta}}\right) \right|$$

$$\psi = \phi + \angle G\left(\mathrm{e}^{\mathrm{j}\bar{\theta}}\right)$$

The frequency response of a system whose transfer function is $G(z)$ is

$$G\left(\mathrm{e}^{\mathrm{j}\theta}\right) \qquad \theta \in [0, \pi]$$

a complex function of the real variable θ.

Remember that the frequency response can be defined for stable and unstable LTI systems.