Control of Mobile Robots

Prof. Bascetta

January 15, 2020

SOLUTION

Control of Mobile Robots
Prof. Luca Bascetta

EXERCISE 1

1. A mobile robot is characterised by k kinematic constraints, that are expressed in Pfaffian form as $A^T(\mathbf{q})\,\dot{\mathbf{q}} = \mathbf{0}$, where $\mathbf{q} \in \mathbb{R}^n$ is the configuration vector. Illustrate all the steps of the procedure that allows to derive the kinematic model of the robot, and write the general expression of the kinematic model.

   Denoting by
   $$\{\mathbf{g}_1(\mathbf{q}), \mathbf{g}_2(\mathbf{q}), \dots, \mathbf{g}_{n-k}(\mathbf{q})\}$$
   a basis of $\mathrm{Null}\left(A^T(\mathbf{q})\right)$, the admissible trajectories of the robot are solutions of the nonlinear dynamic system
   $$\dot{\mathbf{q}} = \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{q})\, u_j = G(\mathbf{q})\,\mathbf{u} \qquad m = n - k$$
   We thus call this dynamic system kinematic model of the robot.

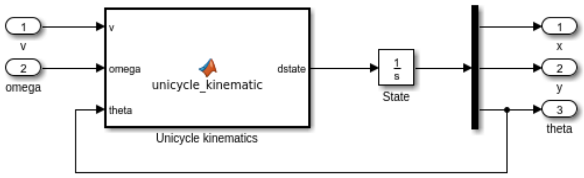
2. Perform all the steps described in the previous item, from the definition of the configuration vector and the $A(\mathbf{q})$ matrix to the expression of the kinematic model, for a unicycle robot.

   The configuration of a unicycle robot is described by the position of the wheel contact point and the wheel orientation, i.e., $\mathbf{q} = \begin{bmatrix} x & y & \theta \end{bmatrix}$.

   The robot is characterised by one nonholonomic constraint, the pure rolling constraint, whose expression in Pfaffian form is
   $$A^T(\mathbf{q})\,\dot{\mathbf{q}} = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix} \dot{\mathbf{q}} = \mathbf{0}$$
   A basis of $\mathrm{Null}\left(A^T(\mathbf{q})\right)$ is given by the two vectors
   $$\mathbf{g}_1(\mathbf{q}) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \qquad \mathbf{g}_2(\mathbf{q}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
   The kinematic model can be thus expressed as
   $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$
   where $u_1 = v$ and $u_2 = \omega$ are the linear and angular velocity of the robot.

3. Draw the Simulink diagram required to simulate the kinematic model of a unicycle robot. If you make use of user-defined functions include the code of each function.

   A possible Simulink diagram to simulate the kinematic model of a unicycle robot is shown in the picture below.

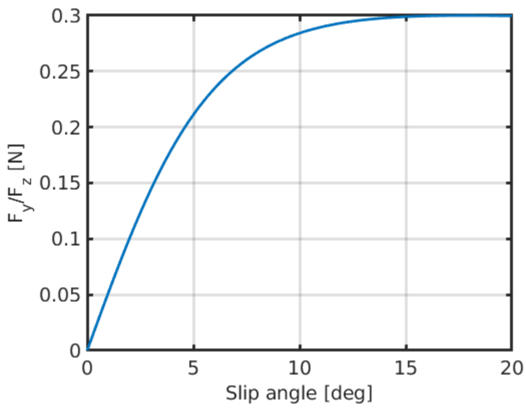
   The user-defined function unicycle_kinematic is composed of the following lines of code:

```
function dstate = unicycle_kinematic(v,omega,theta)

dx     = v*cos(theta);
dy     = v*sin(theta);
dtheta = omega;

dstate = [dx, dy, dtheta];
```

EXERCISE 2

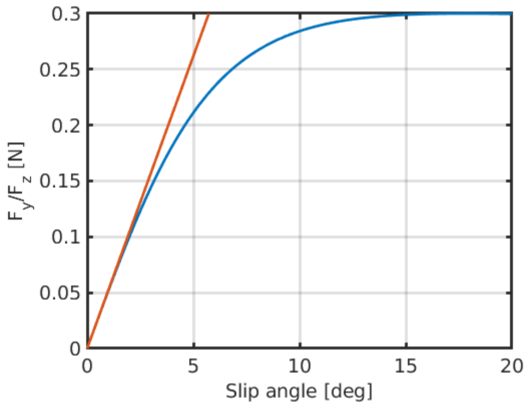
1. As a result of an experimental campaign performed on snow, a tire lateral force has been characterised interpolating experimental data with the following Pacejka Magic Formula (whose plot is shown in the picture below)

$$\frac{F_y}{F_z} = 0.3 \sin\left(2 \arctan\left(5\alpha - \left(5\alpha - \arctan\left(5\alpha\right)\right)\right)\right)$$



   Determine, using the Magic Formula and motivating the result, the value of the cornering stiffness, and draw on the picture the cornering stiffness approximation of the lateral force/slip relation.

   From the magic formula it follows that $B = 5$, $C = 2$, and $D = 0.3$. As a consequence the cornering stiffness is given by $BCD = 3$ Ns/rad.
   The cornering stiffness approximation is shown in the picture below.



2. The cornering stiffness approximation cannot represent the tire force saturation. Illustrate a modelling approach (different from the Pacejka Magica Formula) that allows to represent the saturation and is suitable for model-based control.

   A well-known modelling approach that is suitable for model-based control and allows to represent the saturation is Fiala tire model

$$F_y = \begin{cases} C_\alpha z \left(-1 + \frac{|z|}{z_{sl}} - \frac{z^2}{3z_{sl}^2} \right) & |z| < z_{sl} \\ -\mu F_z \text{sign}\left(\alpha\right) & |z| \geq z_{sl} \end{cases}$$

   where α is the slip angle, C_α the cornering stiffness, $z = \tan\left(\alpha\right)$, and z_{sl} is the minimum value of z that gives full sliding.

3. During a curve the same tire is characterised by a slip angle of 5 deg. What is the corresponding value of the lateral force F_y, assuming $F_z = 150$ N? What is the maximum longitudinal force F_x the tire can generate in these conditions?

   From the Pacejka Magic Formula (or the picture) it follows that $\mu = 0.3$, as a consequence the maximum force the tire can generate is $\mu F_z = 45$ N.
   When $\alpha = 0.0873$ rad the lateral force is equal to

$$F_y = 150\left(0.3 \sin\left(2 \arctan\left(5\alpha - \left(5\alpha - \arctan\left(5\alpha\right)\right)\right)\right)\right) = 31.67 \text{ N}$$

   According to the friction circle constraint, the maximum value of the longitudinal force F_x is given by

$$F_x = \sqrt{\mu^2 F_z^2 - F_y^2} = 31.97 \text{ N}$$

EXERCISE 3

1. Write and explain the pseudocode of the algorithm to construct the probabilistic roadmap used by PRM.

   The PRM algorithm to construct the roadmap follows:

   $V \leftarrow \emptyset$;
   $E \leftarrow \emptyset$;
   for $i = 1, \ldots, N$ **do**
     $\mathbf{q}_{rand} \leftarrow SampleFree_i$;
     $U \leftarrow Near\left(G, \mathbf{q}_{rand}, r\right)$;
     $V \leftarrow V \cup \{\mathbf{q}_{rand}\}$;
     foreach $\mathbf{u} \in U$ *in order of increasing* $\|\mathbf{u} - \mathbf{q}_{rand}\|$ **do**
       if $\mathbf{q}_{rand}$ *and* $\mathbf{u}$ *are not in the same connected component of* G **then**
         if $CollisionFree\left(\mathbf{q}_{rand}, \mathbf{u}\right)$ **then**
           $E \leftarrow E \cup \{(\mathbf{q}_{rand}, \mathbf{u})\}$;
         end
       end
     end
   end
   return $G = (V, E)$

   N vertex are sampled from the free space, then for each vertex $\mathbf{q}_{rand}$ the set of near nodes, i.e., the set of nodes in a ball of radius r centred in $\mathbf{q}_{rand}$, is computed and all the collision free connections between $\mathbf{q}_{rand}$ and the nodes in the near node set that are not in the same connected component are generated.

2. Explain how the previous algorithm has to be modified in order to obtain sPRM and PRM* algorithms.

   The only difference between sPRM and PRM is that sPRM connects all the nodes in the near node set, without checking if they are in the same connected component.
   The sPRM algorithm to construct the roadmap follows:

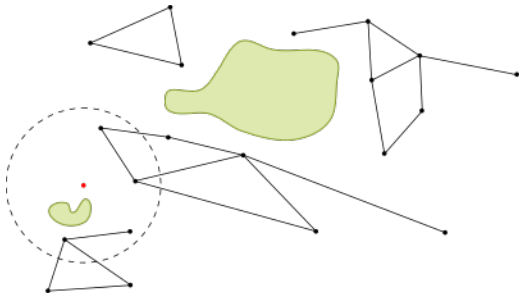
   $V \leftarrow \{\mathbf{q}_{init}\} \cup \{SampleFree_i, i = 1, \ldots, N\}$;
   $E \leftarrow \emptyset$;
   foreach $\mathbf{v} \in V$ **do**
     $U \leftarrow Near\left(G, \mathbf{v}, r\right) \setminus \{\mathbf{v}\}$;
     foreach $\mathbf{u} \in U$ **do**
       if $CollisionFree\left(\mathbf{v}, \mathbf{u}\right)$ **then**
         $E \leftarrow E \cup \{(\mathbf{v}, \mathbf{u})\}$;
       end
     end
   end
   return $G = (V, E)$

   The only difference between PRM and PRM* is in the way the near set is computed. In PRM* the radius of the near neighbourhood is related to the number of sampled nodes, i.e.,

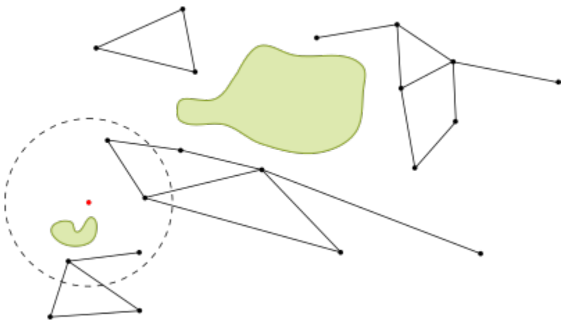
   $$U \leftarrow Near\left(G, \mathbf{q}_{rand}, \gamma_{PRM}\left(\log\left(N\right)/N\right)^{1/d}\right) \setminus \{\mathbf{q}_{rand}\}$$

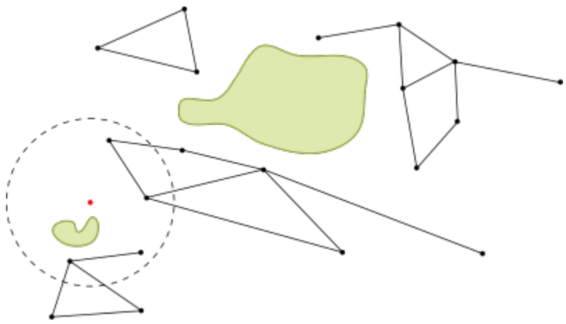
where d is the dimension of the configuration space, and γ_{PRM} is a suitable constant.

3. After the execution of some iterations of PRM algorithm we have the situation depicted below

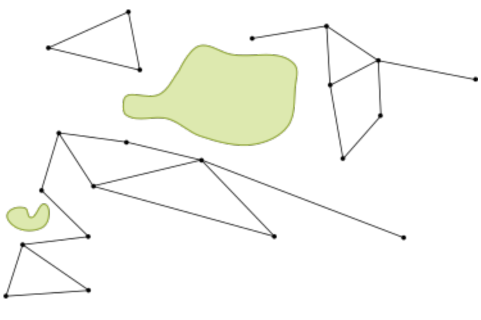


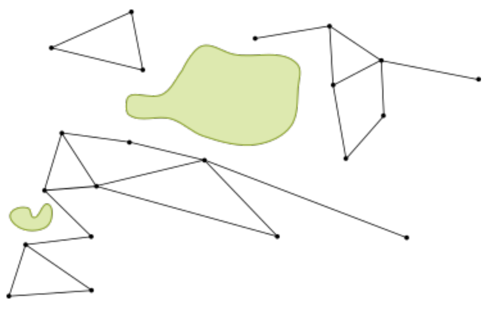
where the black dots are the nodes sampled in previous iterations, the red dot is the node sampled in the actual iteration, the green blobs are obstacles and the black dashed circle is the neighbourhood considered to defined the Near set.
Draw in the left picture the result of an iteration of PRM and in the right picture the result of an iteration of sPRM.





The result of an iteration of PRM (left picture) and of sPRM (right picture) is shown in the pictures below





EXERCISE 4

1. What is the *canonical simplified model for nonholonomic mobile robots*? Why is it important in the context of designing a controller for a nonholonomic robot?

The canonical simplified model is a way to unify in a single model the unicycle, differential drive and bicycle kinematic models, in order to apply the same controller design procedure to the different kinematic models. The canonical simplified model has the following expression

$$\dot{x} = v\cos\theta$$
$$\dot{y} = v\sin\theta$$
$$\dot{\theta} = \omega$$

2. Show how the unicycle, differential drive, and rear-wheel drive bicycle kinematic models can be made equivalent to the canonical model.

The canonical model is the model of a unicycle robot.
The differential drive model can be reduced to the unicycle model applying the following transformation

$$\omega_R = \frac{v + \omega d/2}{r} \qquad \omega_L = \frac{v - \omega d/2}{r}$$

where ω_R and ω_L are the right and left wheel velocities, r is the wheel radius, and d is the distance between left and right wheel contact point.
For the rear-wheel drive bicycle we need to assume that the steering rate limit is so high that the steering angle can be changed instantaneously, then the bicycle model can be reduced to the unicycle model applying the following transformation

$$v = v \qquad \phi = \arctan\left(\frac{\omega\ell}{v}\right)$$

where ϕ is the steering angle and ℓ the length of the bicycle.

3. Consider a bicycle kinematic model without reverse. Show how the actuation constraints $0 \le v \le v_M$ and $-\phi_M \le \phi \le \phi_M$ can be rewritten in terms of the canonical model input variables.

Considering the transformation for the bicycle

$$v = v \qquad \phi = \arctan\left(\frac{\omega\ell}{v}\right)$$

and assuming that $\arctan(x) \approx x$, we have

$$0 \le v \le v_M \qquad -\phi_M \le \frac{\omega\ell}{v} \le \phi_M$$

The second constraint can be rewritten as two separate constraints as follows

$$\omega \le \frac{\phi_M}{\ell}v \qquad \omega \ge -\frac{\phi_M}{\ell}v$$